Exercise Problem Sets 9

May. 2. 2021

Problem 1. Use the Fourier series of the function $f : (-\pi, \pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & -\pi < x < 0\,, \\ \pi - x & 0 \leqslant x < \pi\,, \end{cases}$$

and compute

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^4}\,.$$

Solution. From Problem 1 in Exercise 8, we find that

$$s_k = \frac{1}{k} \quad \forall\, k \in \mathbb{N}\,, \qquad c_k = \frac{1-(-1)^k}{k^2\pi} \quad \forall\, k \in \mathbb{N} \qquad \text{and} \qquad c_0 = \frac{\pi}{2}\,.$$

Since

$$\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx = \int_0^{\pi} (\pi - x)^2\,dx = -\frac{1}{3}(\pi - x)^3\Big|_{x=0}^{x=\pi} = \frac{\pi^3}{3}$$

and

$$\sum_{k=1}^{\infty} s_k^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\,, \qquad \sum_{k=1}^{\infty} c_k^2 = \sum_{k=1}^{\infty} \frac{2^2}{(2k-1)^4\pi^2} = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\,,$$

the Parseval identity implies that

$$\frac{\pi^2}{6} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x)\big|^2\,dx = \frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\big(c_k^2 + s_k^2\big) = \frac{\pi^2}{16} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} + \frac{\pi^2}{12}\,.$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{\pi^2}{2}\Big(\frac{\pi^2}{6} - \frac{\pi^2}{12} - \frac{\pi^2}{16}\Big) = \frac{\pi^4}{96}\,.$$

Note that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \sum_{k=1}^{\infty}\frac{1}{(2k)^4} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4} + \frac{\pi^4}{96}\,;$$

thus rearranging terms we conclude that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}\,.$$

□

Problem 2. This problem contributes to another proof of showing that the Fourier series of f converges uniformly to f on $\mathbb{R}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for $\frac{1}{2} < \alpha \leqslant 1$. Complete the following.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic such that f is Riemann integrable on $[-\pi, \pi]$. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x+h) - f(x-h)\big|^2\,dx = \sum_{k=-\infty}^{\infty} 4\sin^2(kh)|\widehat{f}_k|^2\,.$$

Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies

$$\sum_{k=-\infty}^{\infty} \sin^2(kh)|\widehat{f}_k|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} 2^{2(\alpha-1)}|h|^{2\alpha} . \tag{$\star$}$$

Hint: Find the Fourier series of $g(x) = f(x+h) - f(x-h)$ for given $h > 0$ and make use of the Parseval identity.

2. Let $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, and $p \in \mathbb{N}$. Show that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{2\alpha}}{2^{2\alpha p+1}} .$$

Hint: Let $h = \dfrac{\pi}{2^{p+1}}$ in ($\star$).

3. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\frac{1}{2} < \alpha \leqslant 1$, then $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}_k| < \infty$; thus Problem 6 of Exercise 7 implies that the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof. 1. For $h \neq 0$, let $g(x) = f(x+h) - f(x-h)$. Then by substitution of variables,

$$\begin{aligned}
\widehat{g}_k &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(y+h)e^{-iky}\,dy - \int_{-\pi}^{\pi} f(y-h)e^{-iky}\,dy\Big] \\
&= \frac{1}{2\pi}\Big[\int_{-\pi+h}^{\pi+h} f(x)e^{-ikx}e^{ikh}\,dx - \int_{-\pi-h}^{\pi-h} f(x)e^{-ikx}e^{-ikh}\,dx\Big]
\end{aligned}$$

so that the periodicity of f and the function $y = e^{-ikx}$ implies that

$$\begin{aligned}
\widehat{g}_k &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(x)e^{-ikx}e^{ikh}\,dx - \int_{-\pi-h}^{\pi-h} f(x)e^{-ikx}e^{-ikh}\,dx\Big] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\big(e^{ikh} - e^{-ikh}\big)\,dx = \frac{2i\sin(kh)}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = 2i\sin(kh)\widehat{f}_k .
\end{aligned}$$

Therefore, the Parseval identity shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x+h) - f(x-h)\big|^2\,dx = \sum_{k=-\infty}^{\infty} \big|\widehat{g}_k\big|^2 = \sum_{k=-\infty}^{\infty} 4\sin^2(kh)\big|\widehat{f}_k\big|^2 .$$

If in addition $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, then the identity above implies that

$$\sum_{k=-\infty}^{\infty} 4\sin^2(kh)\big|\widehat{f}_k\big|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}(2h)^{2\alpha}\,dx = \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}(2h)^{2\alpha}$$

which verifies ($\star$).

2. For each $p \in \mathbb{N}$, letting $h = \dfrac{\pi}{2^{p+1}}$ in ($\star$) we find that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} \sin^2\frac{k\pi}{2^{p+1}}\big|\widehat{f}_k\big|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} 2^{2(\alpha-1)} \frac{\pi^{2\alpha}}{2^{2(p+1)\alpha}} = \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^{2\alpha}}{2^{2(p\alpha+1)}}$$

Since for $2^{p-1} \leqslant |k| < 2^p$, $\sin^2 \dfrac{k\pi}{2^{p+1}} \geqslant \dfrac{1}{2}$, the inequality above implies that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{2\alpha}}{2^{2p\alpha+1}}\,.$$

3. Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0.5, 1]$. For each $p \in \mathbb{N}$, by the Cauchy inequality and the result in part 3 we obtain that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k| \leqslant \Big(\sum_{2^{p-1} \leqslant |k| < 2^p} 1 \Big)^{\frac{1}{2}} \Big(\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \Big)^{\frac{1}{2}} = \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{\alpha}}{2^{p(\alpha - \frac{1}{2})+1}}\,.$$

Therefore, by the fact that $\sum\limits_{p=1}^{\infty} \dfrac{1}{2^{p(\alpha-\frac{1}{2})}} < \infty$ (since $\alpha > \dfrac{1}{2}$), we conclude that

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k| = |\widehat{f}_0| + \sum_{p=1}^{\infty} \sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k| \leqslant |\widehat{f}_0| + \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{\alpha}}{2} \sum_{p=1}^{\infty} \frac{1}{2^{p(\alpha-\frac{1}{2})}} < \infty\,;$$

thus Problem 6 of Exercise 7 implies that the Fourier series of f converges uniformly to f on $\mathbb{R}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0.5, 1]$. □

Problem 3. Let $f : [-L, L] \to \mathbb{R}$ be square integrable; that is, f is Riemann measurable and $\displaystyle\int_{-L}^{L} f(x)^2\,dx < \infty$. Show that if $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficient of f, then

$$\frac{1}{2L} \int_{-L}^{L} f(x)^2\,dx = \frac{c_0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} (c_k^2 + s_k^2)\,.$$

Proof. Define $g(x) = f\big(\frac{Lx}{\pi}\big)$. Then $g : [-\pi, \pi] \to \mathbb{R}$ belongs to $L^2(\mathbb{T})$; thus the Parseval identity shows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)^2\,dx = \frac{\bar{c}_0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} (\bar{c}_k^2 + \bar{s}_k^2)\,,$$

where $\{\bar{c}_k\}_{k=0}^{\infty}$ and $\{\bar{s}_k\}_{k=1}^{\infty}$ are Fourier coefficients of g given by

$$\bar{c}_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos kx\,dx = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L}\,dx = c_k\,,$$
$$\bar{s}_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin kx\,dx = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L}\,dx = s_k\,.$$

The result the follows from the fact that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)^2\,dx = \frac{1}{2L} \int_{-L}^{L} f(x)^2\,dx\,.$$ □

Problem 4. Let $f : [0, L] \to \mathbb{R}$ be a square integrable function.

1. Suppose that $\dfrac{c_0}{2} + \sum\limits_{k=1}^{\infty} c_k \cos \dfrac{k\pi x}{L}$ is the cosine series of f. Find $\sum\limits_{k=1}^{\infty} c_k^2$ in terms of integrals of f and f^2.

2. Suppose that $\sum_{k=1}^{\infty} s_k \cos \frac{k\pi x}{L}$ is the sine series of f. Find $\sum_{k=1}^{\infty} s_k^2$ in terms of integrals of f^2.

Solution. 1. Let $\bar{f}$ be the even extension of f. By the definition of the cosine series of f,

$$s(\bar{f}, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos \frac{k\pi x}{L},$$

where

$$c_k = \frac{1}{L} \int_{-L}^{L} \bar{f}(x) \cos \frac{k\pi x}{L}\, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{k\pi x}{L}\, dx .$$

By Problem 3,

$$\frac{1}{2L} \int_{-L}^{L} \bar{f}(x)^2\, dx = \frac{c_0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} c_k^2 ;$$

thus

$$\sum_{k=1}^{\infty} c_k^2 = \frac{1}{L} \int_{-L}^{L} \bar{f}(x)^2\, dx - \frac{c_0^2}{2} = \frac{2}{L} \int_{0}^{L} f(x)^2\, dx - \frac{2}{L^2} \Big(\int_{0}^{L} f(x)\, dx \Big)^2 .$$

2. Let $\bar{f}$ be the odd extension of f. By the definition of the cosine series of f,

$$s(\bar{f}, x) = \sum_{k=1}^{\infty} s_k \sin \frac{k\pi x}{L},$$

where

$$s_k = \frac{1}{L} \int_{-L}^{L} \bar{f}(x) \sin \frac{k\pi x}{L}\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{k\pi x}{L}\, dx .$$

By Problem 3,

$$\frac{1}{2L} \int_{-L}^{L} \bar{f}(x)^2\, dx = \frac{1}{2} \sum_{k=1}^{\infty} s_k^2 ;$$

thus

$$\sum_{k=1}^{\infty} s_k^2 = \frac{1}{L} \int_{-L}^{L} \bar{f}(x)^2\, dx = \frac{2}{L} \int_{0}^{L} f(x)^2\, dx . \qquad \square$$