Exercise Problem Sets 8

Apr. 24. 2021

Problem 1. Compute the Fourier series of the function $f : (-\pi, \pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & -\pi < x < 0\,, \\ \pi - x & 0 \leqslant x < \pi\,, \end{cases}$$

and show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}\,. \tag{$\star$}$$

Also use the Fourier series of the function $y = x^2$ (which we talked about in class)

$$s(x^2, x) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx$$

to conclude ($\star$).

Solution. We compute the Fourier coefficients as follows. For $k \in \mathbb{N}$,

$$s_k = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin(kx)\,dx = \frac{1}{\pi}\Big[\frac{-(\pi - x)\cos(kx)}{k}\Big|_{x=0}^{x=\pi} - \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\Big] = \frac{1}{k}$$

and

$$\begin{aligned} c_k &= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos(kx)\,dx = \frac{1}{\pi}\Big[\frac{(\pi - x)\sin(kx)}{k}\Big|_{x=0}^{x=\pi} + \frac{1}{k}\int_0^{\pi}\sin(kx)\,dx\Big] \\ &= \frac{-\cos(kx)}{k^2\pi}\Big|_{x=0}^{x=\pi} = \frac{1-(-1)^k}{k^2\pi}\,, \end{aligned}$$

while

$$c_0 = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx = \frac{\pi}{2}\,.$$

Therefore, by the fact that $\lim\limits_{x\to 0^-} f(x) = 0$ and $\lim\limits_{x\to 0^+} f(x) = \pi$,

$$\frac{\pi}{4} + \sum_{k=1}^{\infty}\Big(\frac{1-(-1)^k}{k^2\pi}\cos(kx) + \frac{1}{k}\sin(kx)\Big) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0\,, \\ \pi - x & \text{if } 0 < x \leqslant \pi\,, \\ \dfrac{\pi}{2} & \text{if } x = 0\,. \end{cases}$$

We note that the case $x = 0$ implies that

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k^2\pi}$$

which shows the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}\,.$$

We also note that the identity above can be obtained by

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}+\sum_{k=1}^{\infty}\frac{1}{(2k)^2}=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}+\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}$$

so that

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{3}{4}\frac{\pi^2}{6}=\frac{\pi^2}{8}\,.$$

□

Problem 2. In class we prove that the theorem

> Let $f:\mathbb{R}\to\mathbb{R}$ be $2L$-periodic piecewise Hölder continuous with exponent $\alpha\in(0,1]$. Then
>
> $$\lim_{n\to\infty}s_n(f,x_0)=\frac{f(x_0^+)+f(x_0^-)}{2}\qquad\forall\,x_0\in\mathbb{R}\,.$$
>
> Moreover, if x_0 is a jump discontinuity of f and
>
> $$f(x_0^+)-f(x_0^-)=a\neq 0\,,$$
>
> then there exists a constant $c=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\frac{\sin x}{x}\,dx-\frac{1}{2}\approx 0.089490$ such that
>
> $$\lim_{n\to\infty}s_n\big(f,x_0\pm\frac{L}{n}\big)=f(x_0^{\pm})\pm ca\,.$$

holds for $L=\pi$. Use this fact to prove that the theorem holds for general $L>0$.

Proof. Suppose that the theorem holds for the case $L=\pi$. Let $f:\mathbb{R}\to\mathbb{R}$ be $2L$-periodic piecewise Hölder continuous with exponent $\alpha\in(0,1]$. Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=f\big(\frac{Lx}{\pi}\big)$ (or equivalently, $f(x)=g\big(\frac{\pi x}{L}\big)$). Then g is 2π-periodic piecewise Hölder continuous exponent $\alpha\in(0,1]$, and

$$s_n(g,x)=s_n\big(f,\frac{Lx}{\pi}\big)\qquad\text{and}\qquad s_n(f,x)=s_n\big(g,\frac{\pi x}{L}\big)\,.$$

Let $y_0=\dfrac{\pi x_0}{L}$. By the fact that $\displaystyle\lim_{x\to x_0^{\pm}}h(cx)=\lim_{y\to(cx_0)^{\pm}}h(x)$ if $c>0$,

$$\begin{aligned}\lim_{n\to\infty}s_n(f,x_0)=\lim_{n\to\infty}s_n(g,y_0)&=\frac{1}{2}\Big[\lim_{y\to y_0^+}g(y)+\lim_{y\to y_0^-}g(y)\Big]\\&=\frac{1}{2}\Big[\lim_{x\to x_0^+}g\big(\frac{\pi x}{L}\big)+\lim_{x\to x_0^-}g\big(\frac{\pi x}{L}\big)\Big]=\frac{1}{2}\Big[\lim_{x\to x_0^+}f(x)+\lim_{x\to x_0^-}f(x)\Big]=\frac{f(x_0^+)+f(x_0^-)}{2}\,.\end{aligned}$$

Moreover, if x_0 is a jump discontinuity of f, then y_0 is a jump discontinuity of g so that

$$\begin{aligned}\lim_{n\to\infty}s_n\big(f,x_0+\frac{L}{n}\big)&=\lim_{n\to\infty}s_n\big(g,\frac{\pi}{L}\big(x_0+\frac{L}{n}\big)\big)=\lim_{n\to\infty}s_n\big(g,y_0+\frac{\pi}{n}\big)\\&=\lim_{y\to y_0^+}g(y)+c\Big[\lim_{y\to y_0^+}g(y)-\lim_{y\to y_0^-}g(y)\Big]\\&=\lim_{x\to x_0^+}g\big(\frac{\pi x}{L}\big)+c\Big[\lim_{x\to x_0^+}g\big(\frac{\pi x}{L}\big)-\lim_{x\to x_0^-}g\big(\frac{\pi x}{L}\big)\Big]=f(x_0^+)+ca\,.\end{aligned}$$

Similarly, $\displaystyle\lim_{n\to\infty}s_n\big(f,x_0+\frac{L}{n}\big)=f(x_0^-)-ca$. □

Problem 3. For a given function $f : [0, L] \to \mathbb{R}$, the even extension of f is a function $\bar{f} : [-L, L] \to \mathbb{R}$ such that

$$\bar{f}(x) = f(-x) \qquad \forall\, x \in [-L, 0)\,.$$

1. Let $f : [0, L] \to \mathbb{R}$ be an integrable function. The cosine series of f is the Fourier series of the even extension of f. Find the cosine series of f.

2. Suppose in addition $f : [0, L] \to \mathbb{R}$ is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$. Show that the cosine series of f at $x_0 \in (0, L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$.

Proof. 1. Let $\bar{f}$ be the even extension of f, and $\{c_k\}_{k=0}^{\infty}$, $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients of $\bar{f}$. Then by the fact that $\bar{f}$ is even, $s_k = 0$ for all $k \in \mathbb{N}$. Moreover,

$$\begin{aligned} c_k &= \frac{1}{L}\int_{-L}^{L} \bar{f}(x)\cos\frac{k\pi x}{L}\,dx = \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{k\pi x}{L}\,dx + \frac{1}{L}\int_{-L}^{0} f(-x)\cos\frac{k\pi x}{L}\,dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{k\pi x}{L}\,dx + \frac{1}{L}\int_{L}^{0} f(x)\cos\frac{k\pi(-x)}{L} d(-x) \\ &= \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{k\pi x}{L}\,dx\,. \end{aligned}$$

Therefore, the cosine series of f is

$$s(\bar{f}, x) = \frac{1}{L}\int_{0}^{L} f(x)\,dx + \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_{0}^{L} f(y)\cos\frac{k\pi y}{L}\,dy\Big)\cos\frac{k\pi x}{L}\,.$$

2. If f is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$, then the odd extension $\bar{f}$ of f is also piecewise Hölder continuous with exponent $\alpha \in (0, 1]$; thus

$$s(\bar{f}, x_0) = \frac{\bar{f}(x_0^+) + \bar{f}(x_0^-)}{2} = \frac{f(x_0^+) + f(x_0^-)}{2}$$

which shows that the cosine series of f at $x_0 \in (0, L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$. □

Problem 4. For a given function $f : [0, L] \to \mathbb{R}$, the odd extension of f is a function $\bar{f} : [-L, L] \to \mathbb{R}$ such that

$$\bar{f}(x) = -f(-x) \qquad \forall\, x \in [-L, 0)\,.$$

1. Let $f : [0, L] \to \mathbb{R}$ be an integrable function. The sine series of f is the Fourier series of the odd extension of f. Find the cosine series of f.

2. Suppose in addition $f : [0, L] \to \mathbb{R}$ is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$. Show that the sine series of f at $x_0 \in (0, L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$.

Proof. 1. Let $\bar{f}$ be the odd extension of f, and $\{c_k\}_{k=0}^\infty$, $\{s_k\}_{k=1}^\infty$ be the Fourier coefficients of $\bar{f}$. Then by the fact that $\bar{f}$ is odd, $c_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Moreover,

$$\begin{aligned} s_k &= \frac{1}{L}\int_{-L}^{L} \bar{f}(x)\sin\frac{k\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx - \frac{1}{L}\int_{-L}^{0} f(-x)\sin\frac{k\pi x}{L}\,dx \\ &= \frac{1}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx - \frac{1}{L}\int_L^0 f(x)\sin\frac{k\pi(-x)}{L}d(-x) \\ &= \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx\,. \end{aligned}$$

Therefore, the sine series of f is

$$s(\bar{f},x) = \frac{2}{L}\sum_{k=1}^\infty \Big(\int_0^L f(y)\sin\frac{k\pi y}{L}\,dy\Big)\sin\frac{k\pi x}{L}\,.$$

2. If f is piecewise Hölder continuous with exponent $\alpha \in (0,1]$, then the odd extension $\bar{f}$ of f is also piecewise Hölder continuous with exponent $\alpha \in (0,1]$; thus

$$s(\bar{f},x_0) = \frac{\bar{f}(x_0^+) + \bar{f}(x_0^-)}{2} = \frac{f(x_0^+) + f(x_0^-)}{2}$$

which shows that the sine series of f at $x_0 \in (0,L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$. □

Problem 5. Find the cosine series and the sine series of the function $f : [0,\pi] \to \mathbb{R}$ given by $f(x) = \dfrac{1}{2\pi}(x-\pi)$.

Solution. 1. For $k \neq 0$,

$$\begin{aligned} \frac{2}{\pi}\int_0^\pi \frac{1}{2\pi}(x-\pi)\cos kx\,dx &= \frac{1}{\pi^2}\int_0^\pi x\cos kx\,dx = \frac{1}{\pi^2}\Big[\frac{x\sin kx}{k}\Big|_{x=0}^{x=\pi} - \frac{1}{k}\int_0^\pi \sin kx\,dx\Big] \\ &= \frac{1}{k^2\pi^2}\cos kx\Big|_{x=0}^{x=\pi} = \frac{(-1)^k-1}{k^2\pi^2} \end{aligned}$$

while

$$\frac{2}{\pi}\int_0^\pi \frac{1}{2\pi}(x-\pi)\,dx = \frac{1}{2\pi^2}(x-\pi)^2\Big|_{x=0}^{x=\pi} = -\frac{1}{2}\,.$$

Therefore, the cosine series of f is

$$-\frac{1}{4} - \frac{2}{\pi^2}\sum_{k=1}^\infty \frac{1}{(2k-1)^2}\cos(2k-1)x\,.$$

2. For $k \neq 0$,

$$\begin{aligned} \frac{2}{\pi}\int_0^\pi \frac{1}{2\pi}(x-\pi)\sin kx\,dx &= \frac{1}{\pi^2}\int_0^\pi (x-\pi)\sin kx\,dx \\ &= \frac{1}{\pi^2}\Big[\frac{-(x-\pi)\cos kx}{k}\Big|_{x=0}^{x=\pi} + \frac{1}{k}\int_0^\pi \cos kx\,dx\Big] = \frac{-1}{k\pi}\,. \end{aligned}$$

Therefore, the sine series of f is

$$-\frac{1}{\pi}\sum_{k=1}^\infty \frac{\sin kx}{k}\,.$$

□