Exercise Problem Sets 6

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Problem 1. Define B to be the set of all even functions in the space $\mathscr{C}([-1,1];\mathbb{R})$; that is, $f \in B$ if and only if f is continuous on $[-1,1]$ and $f(x) = f(-x)$ for all $x \in [-1,1]$. Prove that B is closed but not dense in $\mathscr{C}([-1,1];\mathbb{R})$. Hence show that even polynomials are dense in B, but not in $\mathscr{C}([-1,1];\mathbb{R})$.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in B and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $[-1,1]$. Then f is continuous. Moreover, for each $x \in [-1,1]$,

$$f(x) = \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} f_k(-x) = f(-x)\,;$$

thus f is even. Therefore, $f \in B$ which shows that B is closed. However, B is not dense in B since there exists no $f \in B$ satisfying that

$$\max_{x\in[-1,1]} \big|f(x) - x\big| < \frac{1}{2}$$

since

$$\max_{x\in[-1,1]} \big|f(x) - x\big| \geqslant \max\{\big|f(1)-1\big|, \big|f(-1)+1\big|\} = \max\{\big|f(1)-1\big|, \big|f(1)+1\big|\} \geqslant 1\,.$$

Let $\mathcal{A}$ denote the collection of even polynomials, and f be an even continuous function. Then the Weierstrass Theorem implies that there exists a sequence of polynomial $\{p_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \max_{x\in[0,1]} \big|f(\sqrt{x}) - p_n(x)\big| = 0\,.$$

For each $n \in \mathbb{N}$, define $q_n : [-1,1] \to \mathbb{R}$ by $q_n(x) = p_n(x^2)$. Then $\{q_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ and

$$\lim_{n\to\infty} \max_{x\in[-1,1]} \big|f(x) - q_n(x)\big| = \lim_{n\to\infty} \max_{x\in[0,1]} \big|f(x) - p_n(x^2)\big| = \lim_{n\to\infty} \max_{x\in[0,1]} \big|f(\sqrt{x}) - p_n(x)\big| = 0$$

which shows that $\{q_n\}_{n=1}^{\infty}$ converges uniformly to f on $[-1,1]$; thus $\mathcal{A}$ is dense in B. On the other hand, since $\mathcal{A} \subseteq B$, we must have $\bar{\mathcal{A}} \subseteq \bar{B} \subsetneq \mathscr{C}([-1,1];\mathbb{R})$ which implies that $\mathcal{A}$ is not dense in $\mathscr{C}([-1,1];\mathbb{R})$. □

Problem 2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

1. Suppose that

$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$

   Show that $f = 0$ on $[0,1]$.

2. Suppose that for some $m \in \mathbb{N}$,

$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \{0, 1, \cdots, m\}\,.$$

   Show that $f(x) = 0$ has at least $(m+1)$ distinct real roots around which $f(x)$ change signs.

Proof. 1.By the Weierstrass Theorem, for each $k \in \mathbb{N}$ there exists a polynomial p_k such that $\|f - p_k\|_\infty < \frac{1}{k}$. Since $\int_0^1 f(x)x^n\,dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$, we find that

$$\int_0^1 f(x)p_k(x)\,dx = 0 \qquad \forall\, k \in \mathbb{N}\,.$$

Note that $f(f - p_k)$ converges to the zero function uniformly on $[0, 1]$ since

$$\|f(f - p_k)\|_\infty \leqslant \|f\|_\infty \|f - p_k\|_\infty \leqslant \frac{1}{k}\|f\|_\infty \to 0 \;\text{ as }\; k \to \infty\,;$$

thus by the fact that

$$\int_0^1 f(x)^2\,dx = \int_0^1 f(x)\big[f(x) - p_k(x)\big]\,dx\,,$$

we find that $\int_0^1 f(x)^2\,dx = 0$. Therefore, by the continuity of f, we conclude that $f = 0$ on $[0, 1]$.

2. Let

$$D = \Big\{k \in \mathbb{N}\,\Big|\,\text{if } f \in \mathscr{C}([0,1];\mathbb{R}) \text{ and } f \text{ changes signs only around } 0 < \alpha_1 < \cdots < \alpha_k < 1, \text{ then } y = f(x)\prod_{j=1}^{k}(x - \alpha_j) \text{ does not change sign}\Big\}\,.$$

Suppose that $f \in \mathscr{C}([0,1];\mathbb{R})$ changes sign only around $0 < \alpha_1 < 1$. Then $y = f(x)(x - \alpha_1)$ does not change sign so that $1 \in D$. Assume that $k \in D$. If f changes signs only around $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{k+1} < 1$, then the function $y = f(x)(x - \alpha_{k+1})$ changes signs only around $0 < \alpha_1 < \cdots < \alpha_k < 1$; thus $y = f(x)(x - \alpha_{k+1})\prod_{j=1}^{k}(x - \alpha_j) = f(x)\prod_{j=1}^{k+1}(x - \alpha_j)$ does not change sign which shows that $k + 1 \in D$. By induction, we conclude that $D = \mathbb{N}$.

Now suppose the contrary that $f(x) = 0$ has at most m distinct real roots $0 < \alpha_1 < \cdots < \alpha_k < 1$, where $0 \leqslant k \leqslant m$, around which $f(x)$ changes signs. Then $y = f(x)\prod_{j=1}^{k}(x - \alpha_j)$ does not change sign. W.L.O.G., we assume that $f(x)\prod_{j=1}^{k}(x - \alpha_j) \geqslant 0$ for all $x \in [0, 1]$. Then by the fact that

$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \{0, 1, \cdots, m\}\,.$$

and $k \leqslant m$, we find that

$$\int_0^1 f(x)\prod_{j=1}^{k}(x - \alpha_j)\,dx = 0\,;$$

thus the sign-definite property and the continuity of the function $y = f(x)\prod_{j=1}^{k}(x - \alpha_j)$ implies that $f(x)\prod_{j=1}^{k}(x - \alpha_j) = 0$ for all $x \in [0, 1]$. Therefore, $f(x)\prod_{j=1}^{k}(x - \alpha_j) = 0$ for all $x \in [0, 1]\backslash\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ or equivalently, $f(x) = 0$ for all $x \in [0, 1]\backslash\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. The continuity of f further implies that $f = 0$ on $[0, 1]$, a contradiction to that f has at most m distinct real roots around which f changes signs. □

Problem 3. Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\,dx = 0\,.$$

Proof. We first show that $\lim_{n\to\infty} \int_0^1 x^k \sin(nx)\,dx = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Let

$$D = \Big\{ k \in \mathbb{N} \cup \{0\} \,\Big|\, \lim_{n\to\infty} \int_0^1 x^k \sin(nx)\,dx = 0 \Big\}\,.$$

Then $0 \in D$ and $1 \in D$ since

$$\int_0^1 \sin(nx)\,dx = \frac{-\cos(nx)}{n}\Big|_{x=0}^{x=1} = \frac{\cos 0 - \cos n}{n} \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\int_0^1 x\sin(nx)\,dx = \frac{-x\cos(nx)}{n}\Big|_{x=0}^{x=1} + \frac{1}{n}\int_0^1 \cos(nx)\,dx = -\frac{\cos n}{n} + \frac{\sin n}{n^2} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Suppose that $\{0, 1, \cdots, k\} \subseteq D$. Then

$$\begin{aligned}
\int_0^1 x^{k+1}\sin(nx)\,dx &= -\frac{x^{k+1}\cos(nx)}{n}\Big|_{x=0}^{x=1} + \frac{k+1}{n}\int_0^1 x^k \cos(nx)\,dx \\
&= -\frac{\cos n}{n} + \frac{k+1}{n}\Big[\frac{x^k \sin(nx)}{n}\Big|_{x=0}^{x=1} - \frac{k}{n}\int_0^1 x^{k-1}\sin(nx)\,dx\Big] \\
&= -\frac{\cos n}{n} + \frac{(k+1)\sin n}{n^2} - \frac{(k+1)k}{n^2}\int_0^1 x^{k-1}\sin(nx)\,dx \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

By induction, $D = \mathbb{N} \cup \{0\}$.

Having established that $D = \mathbb{N} \cup \{0\}$, we immediately conclude that

$$\lim_{n\to\infty} \int_0^1 p(x)\sin(nx)\,dx = 0 \qquad \text{for all polynomial } p\,.$$

Let $\varepsilon > 0$ be given. By the Weierstrass Theorem, there exists a polynomial p such that $\|f - p\|_\infty < \frac{\varepsilon}{2}$. By the fact that $\lim_{n\to\infty} \int_0^1 p(x)\sin(nx)\,dx = 0$, there exists $N > 0$ such that

$$\Big|\int_0^1 p(x)\sin(nx)\,dx\Big| < \frac{\varepsilon}{2} \quad \textit{whenever} \quad n \geqslant N\,.$$

Therefore, if $n \geqslant N$,

$$\begin{aligned}
\Big|\int_0^1 f(x)\sin(nx)\,dx\Big| &\leqslant \Big|\int_0^1 \big[f(x) - p(x)\big]\sin(nx)\,dx\Big| + \Big|\int_0^1 p(x)\sin(nx)\,dx\Big| \\
&\leqslant \int_0^1 \|f - p\|_\infty\,dx + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

which establishes that $\lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\,dx = 0$. □

Problem 4. Put $p_0 = 0$ and define

$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

Show that $\{p_k\}_{k=1}^\infty$ converges uniformly to $|x|$ on $[-1,1]$.
Hint: Use the identity

$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big]$$

to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that

$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$

if $|x| \leqslant 1$.

Proof. Let $D = \big\{k \in \mathbb{N} \,\big|\, 0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x| \ \forall\, x \in [-1,1]\big\}$. By the fact that $p_1(x) = \dfrac{x^2}{2}$ and $0 \leqslant p_1(x) \leqslant |x|$ for all $x \in [-1,1]$, we find that $p_2(x) = p_1(x) + \dfrac{|x|^2 - p_1(x)^2}{2} \geqslant p_1(x)$. Therefore, $1 \in D$.

Assume that $k \in D$. Then the identity

$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

implies that $p_{k+1}(x) \geqslant p_k(x) \geqslant 0$ on $[-1,1]$. Moreover, using the identity

$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big], \tag{$\star$}$$

we find that if $x \in [-1,1]$,

$$|x| - p_{k+1}(x) \geqslant \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + |x|}{2}\Big] = \big[|x| - p_k(x)\big]\big(1 - |x|\big) \geqslant 0\,;$$

thus $p_{k+1}(x) \leqslant |x|$ on $[-1,1]$. Therefore, $k+1 \in D$ so that $D = \mathbb{N}$ by induction.

Using $(\star)$ again, we find that

$$0 \leqslant |x| - p_k(x) = \big[|x| - p_{k-1}(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big] \leqslant \big[|x| - p_{k-1}(x)\big]\big(1 - \frac{|x|}{2}\big) \qquad \forall\, k \in \mathbb{N}\,;$$

thus

$$\begin{aligned} 0 \leqslant |x| - p_k(x) &\leqslant \big[|x| - p_{k-1}(x)\big]\big(1 - \frac{|x|}{2}\big) \leqslant \big[|x| - p_{k-2}(x)\big]\big(1 - \frac{|x|}{2}\big) \\ &\leqslant \cdots \leqslant \big[|x| - p_0(x)\big]\big(1 - \frac{|x|}{2}\big)^k = |x|\big(1 - \frac{|x|}{2}\big)^k\,. \end{aligned}$$

By the fact that $|x|\big(1 - \frac{|x|}{2}\big)^k \leqslant \dfrac{2}{k+1}$ for all $x \in [-1,1]$, we conclude that

$$\lim_{k\to\infty} \max_{x\in[-1,1]} \big|p_k(x) - |x|\big| = 0$$

which shows that $\{p_k\}_{k=1}^\infty$ converges uniformly to $y = |x|$ on $[-1,1]$. □

Problem 5. Suppose that p_n is a sequence of polynomials converging uniformly to f on $[0,1]$ and f is not a polynomial. Prove that the degrees of p_n are not bounded.

Hint: An Nth-degree polynomial p is uniquely determined by its values at $N+1$ points $x_0, \cdots, x_N$ via Lagrange's interpolation formula

$$p(x) = \sum_{k=0}^{N} \pi_k(x) \frac{p(x_k)}{\pi_k(x_k)},$$

where $\pi_k(x) = (x-x_0)(x-x_1)\cdots(x-x_N)/(x-x_k) = \prod_{\substack{0\leqslant j\leqslant N\\ j\neq k}} (x-x_j)$.

Proof. Suppose the contrary that there exists a sequence of polynomial $\{p_n\}_{k=1}^{\infty}$ which converges uniformly to f on $[0,1]$ and $\deg(p_n) \leqslant N$ for all $n \in \mathbb{N}$. W.L.O.G. we assume that

$$\|p_n - f\|_\infty < 1 \qquad \forall\, n \in \mathbb{N}.$$

Then $\big|p_n(x)\big| \leqslant \|f\|_\infty + 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Since $\deg(p_n) \leqslant N$, using the Lagrange interpolation formula with $x_k = k/N$, we have

$$p_n(x) = \sum_{k=0}^{N} \pi_k(x) \frac{p(x_k)}{\pi_k(x_k)} = \sum_{j=0}^{N} a_{jn} x^j .$$

Let $[N/2]$ denote the largest integer smaller than $N/2$. Note that

$$\big|\pi_k(x_k)\big| = \frac{k}{N}\cdot\frac{k-1}{N}\cdot\cdots\cdot\frac{1}{N}\cdot\frac{1}{N}\cdot\cdots\cdot\frac{N-k}{N} \geqslant \frac{[N/2]!}{N^N}$$

so that

$$\Big|\frac{p(x_k)}{\pi_k(x_k)}\Big| \leqslant \frac{(\|f\|_\infty + 1)N^N}{[N/2]!}.$$

Moreover, $\pi_k(x) = \sum\limits_{j=0}^{N} c_j x^j$ with $|c_j| \leqslant C^N_{[N/2]}$. Therefore,

$$|a_{jn}| \leqslant \frac{(\|f\|_\infty + 1)N^N}{[N/2]!} C^N_{[N/2]}(N+1) \qquad \forall\, 0 \leqslant j \leqslant N \text{ and } n \in \mathbb{N}.$$

In other words, the coefficients of each p_n is bounded by a fixed constant. This allows us to pick a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} a_{jn_k} = a_j \text{ exists for all } 0 \leqslant j \leqslant N.$$

This implies that $\{p_{n_k}\}_{k=1}^{\infty}$ converges uniformly to the polynomial $p(x) = \sum\limits_{j=0}^{N} a_j x^j$ since $\{p_{n_k}\}_{k=1}^{\infty}$ converges pointwise to p and $\{p_n\}_{n=1}^{\infty}$ converges uniformly on $[0,1]$ so that $\{p_{n_k}\}_{k=1}^{\infty}$ converges uniformly on $[0,1]$. On the other hand, since $\{p_n\}_{n=1}^{\infty}$ converges uniformly to f on $[0,1]$, we conclude that $f = p$, a contradiction. □

Problem 6. Consider the set of all functions on $[0,1]$ of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x} \,,$$

where a_j, $b_j \in \mathbb{R}$. Is this set dense in $\mathscr{C}([0,1];\mathbb{R})$?

Proof. Let $\mathcal{A} = \left\{ \sum_{j=1}^{n} a_j e^{b_j x} \,\middle|\, a_j, b_j \in \mathbb{R} \right\}$. Then

1. $\mathcal{A}$ is an algebra since if $f(x) = \sum\limits_{j=1}^{n} a_j e^{b_j x}$ and $g(x) = \sum\limits_{k=1}^{m} c_k e^{d_k x}$, we have

$$\Big(\sum_{j=1}^{n} a_j e^{b_j x}\Big)\big(\sum_{k=1}^{m} c_k e^{d_k x}\Big) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j c_k e^{(b_j+d_k)x} = \sum_{\ell=1}^{N} A_\ell e^{B_\ell x}$$

   for some $A_\ell, B_\ell \in \mathbb{R}$, and clearly, $f+g \in \mathcal{A}$ and $cf \in \mathcal{A}$ if $c \in \mathbb{R}$.

2. $\mathcal{A}$ separates points of $[0,1]$ since the function $f(x) = e^x \in \mathcal{A}$ which is strictly monotone so that $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

3. $\mathcal{A}$ vanishes at no point of $[0,1]$ since the function $f(x) = e^x \in \mathcal{A}$ which is non-zero at every point of $[0,1]$.

By the Stone Theorem, $\mathcal{A}$ is dense in $\mathscr{C}([0,1];\mathbb{R})$. □