# Exercise Problem Sets 5

Mar. 26. 2021

**Problem 1.** Show that the series
$$\sum_{k=1}^{\infty}(-1)^k\frac{x^2+k}{k^2}$$
converges uniformly on every bounded interval.

*Proof.* Since $\sum\limits_{k=1}^{\infty}(-1)^k\frac{1}{k} = -\ln 2$ converges (by the Dirichlet test), we have
$$\sum_{k=1}^{\infty}(-1)^k\frac{x^2+k}{k^2} = \sum_{k=1}^{\infty}(-1)^k\frac{x^2}{k^2} - \ln 2 \qquad \forall\, x \in \mathbb{R}\,.$$

Let $M_k = \frac{R^2}{k^2}$. Then

1. $\sup\limits_{x\in[-R,R]}\left|(-1)^k\frac{x^2}{k^2}\right| \leqslant M_k$ for all $k \in \mathbb{N}$.

2. $\sum\limits_{k=1}^{\infty} M_k < \infty$ (by the integral test).

Therefore, the Weierstrass $M$-test implies that $\sum\limits_{k=1}^{\infty}(-1)^k\frac{x^2}{k^2}$ converges uniformly on $[-R, R]$. □

**Problem 2.** Determine which of the following real series $\sum\limits_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$

2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$

3. $g_k(x) = \dfrac{(-1)^k}{\sqrt{k}}\cos(kx)$ on $\mathbb{R}$.

*Proof.* 1. By the definition of $g_k$, we find that the partial sum $S_n(x) = \sum\limits_{k=1}^{n} g_k(x)$ satisfies that for all $n \in \mathbb{N}$,
$$S_{2n}(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-1, 2n]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$
and
$$S_{2n-1}(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-3, 2n-2] \cup (2n-1, \infty)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ converges pointwise to the function

$$S(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-3, 2n-2] \cup \cdot\,, \\ 0 & \text{otherwise} \end{cases}$$

or more precisely,

$$S(x) = \sum_{k=1}^{\infty} \mathbf{1}_{(2k-1,2k]}(x)\,.$$

The convergence is uniformly on any bounded subset of $\mathbb{R}$, and the limit function $S$ has discontinuities on $\mathbb{N}$.

2. Let $M_k = \dfrac{1}{k^2}$. Then $\sup\limits_{x\in\mathbb{R}} |g_k(x)| \leqslant M_k$ and $\sum\limits_{k=1}^{\infty} M_k$ converges (by the integral test). Therefore, the Weierstrass $M$-test implies that $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $\mathbb{R}$. Since $g_k$ is continuous on $\mathbb{R}$, we find that $\sum\limits_{k=1}^{\infty} g_k$ is continuous on $\mathbb{R}$.

3. If $x = (2n+1)\pi$ for some $n \in \mathbb{Z}$, then $\cos(kx) = (-1)^k$ for all $k \in \mathbb{N}$; thus $\sum\limits_{k=1}^{\infty} g_k(x)$ diverges at $x = (2n+1)\pi$ (by the integral test).

   Now suppose that $x \notin \{(2n+1)\pi \,|\, n \in \mathbb{Z}\}$. Let $S_n(x) = \sum\limits_{k=1}^{n} (-1)^k \cos(kx)$. Then $S_n(x) = \sum\limits_{k=1}^{n} \cos(k(x+\pi))$ and

$$\begin{aligned} 2\sin\frac{x+\pi}{2} S_n(x) &= \sum_{k=1}^{n} \Big[\sin\big(k+\frac{1}{2}\big)(x+\pi) - \sin\big(k-\frac{1}{2}\big)(x+\pi)\Big] \\ &= \sin\big(n+\frac{1}{2}\big)(x+\pi) - \sin\frac{x+\pi}{2}\,; \end{aligned}$$

   thus

$$S_n(x) = \frac{(-1)^n \cos(n+\frac{1}{2})x}{2\cos\frac{x}{2}} - \frac{1}{2} \qquad \forall\, x \in \mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\,.$$

   The equality above shows that

$$|S_n(x)| \leqslant \frac{1}{2|\cos\frac{x}{2}|} + \frac{1}{2} \qquad \forall\, x \in \mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\,,$$

   which is bounded independent of $n$. The Dirichlet test then shows that $\sum\limits_{k=1}^{\infty} g_k(x)$ converges for all $x \in \mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}$. Therefore, $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise on $\mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}$.

   Let $A \subseteq \mathbb{R}$ be a set satisfying that

$$d\big(x, \{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\big) = \inf\big\{|x-y| \,\big|\, y \in \{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\big\} \geqslant \delta \qquad \forall\, x \in A\,.$$

Then the computation above shows that $|S_n(x)| \leqslant R \equiv \dfrac{1}{2|\cos\frac{\delta}{2}|} + \dfrac{1}{2}$ for all $x \in A$. If $n > m$, we have

$$\begin{aligned}
\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx) &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}\big[S_k(x) - S_{k-1}(x)\big] = \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_k(x) - \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_{k-1}(x) \\
&= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_k(x) - \sum_{k=m}^{n-1} \frac{1}{\sqrt{k+1}} S_k(x) \\
&= \frac{1}{\sqrt{n}} S_n(x) - \frac{1}{\sqrt{m+1}} S_m(x) + \sum_{k=m+1}^{n-1} \Big(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\Big) S_k(x)\,;
\end{aligned}$$

thus if $x \in A$,

$$\Big|\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx)\Big| \leqslant \Big[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m+1}} + \sum_{k=m+1}^{n-1} \Big(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\Big)\Big] R = \frac{2R}{\sqrt{m+1}}\,.$$

Therefore, for a given $\varepsilon > 0$, by choosing $N > 0$ satisfying $\dfrac{2R}{\sqrt{N+1}} < \varepsilon$ we conclude that

$$\Big|\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx)\Big| < \varepsilon \quad \text{whenever } n > m \geqslant N \text{ and } x \in A\,.$$

By the Cauchy criterion, $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $A$; thus $\sum\limits_{k=1}^{\infty} g_k$ is continuous at every point at which the series converges. □

**Problem 3.** Suppose that the series $\sum\limits_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that $f$ is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$. Show that

$$S_n(x) = (1-x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$

and $f(x) = (1-x)\sum\limits_{n=0}^{\infty} s_n x^n$.

2. Using the representation of $f$ from above to conclude that $\lim\limits_{x\to 1^-} f(x) = 0$.

3. What if $\sum\limits_{n=0}^{\infty} a_n$ is convergent but not zero?

*Proof.* 1. Let $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

$$\begin{aligned}
S_n(x) &= \sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n} a_k x^k = s_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k \\
&= s_0 + \sum_{k=1}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k = \sum_{k=0}^{n} s_k x^k - \sum_{k=0}^{n-1} s_k x^{k+1} \\
&= s_n x^n + \sum_{k=0}^{n-1} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k \\
&= (1-x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n\,.
\end{aligned}$$

Therefore, by the fact that $\lim_{n\to\infty} s_n = 0$, we find that if $x \in (-1, 1]$,

$$f(x) = \lim_{n\to\infty} S_n(x) = (1-x)\sum_{k=0}^{\infty} s_k x^k .$$

2. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} s_n = 0$, there exists $N > 0$ such that $|s_n| < \frac{\varepsilon}{2}$ for all $n \geqslant N$. Choose $0 < \delta < 1$ such that $\delta \sum_{k=0}^{N-1} |s_k| < \frac{\varepsilon}{2}$. Then if $1 - \delta < x < 1$,

$$\begin{aligned} \big|f(x)\big| &\leqslant |1-x| \sum_{k=0}^{N-1} |s_k||x|^k + |1-x| \sum_{k=N}^{\infty} |s_k||x|^k \\ &\leqslant \delta \sum_{k=0}^{N-1} |s_k| + \frac{\varepsilon}{2}|1-x||x|^N \sum_{k=0}^{\infty} |x|^k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}|1-x|\frac{1}{1-|x|} = \varepsilon . \end{aligned}$$

Therefore, $\lim_{x\to 1^-} f(x) = 0 = f(1)$ which shows that $f$ is continuous at 1.

3. If $s = \sum_{k=0}^{\infty} a_k \neq 0$, we define a new series $\sum_{n=0}^{\infty} b_n x^n$ by $b_0 = a_0 - s$ and $b_n = a_n$ for all $n \in \mathbb{N}$. Then $g(x) = \sum_{n=0}^{\infty} b_n x^n$ also converges for $x \in (-1, 1]$ and satisfies that $g(1) = 0$. Therefore, 1 and 2 imply that $g$ is continuous at 1; thus $\lim_{x\to 1^-} g(x) = 0$. By the fact that $g(x) = f(x) - s$, we conclude that

$$\lim_{x\to 1^-} f(x) = s = \sum_{n=0}^{\infty} a_n = f(1) . \qquad \square$$

**Problem 4.** Construct the function $g(x)$ by letting $g(x) = |x|$ if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and extending $g$ so that it becomes periodic (with period 1). Define

$$f(x) = \sum_{k=1}^{\infty} \frac{g(4^{k-1}x)}{4^{k-1}} .$$

1. Use the Weierstrass $M$-test to show that $f$ is continuous on $\mathbb{R}$.

2. Prove that $f$ is differentiable at no point.

**Hint:** Google Blancmange function!

*Proof.* 1. Since $g$ is periodic with period 1, we find that

$$\sup_{x\in\mathbb{R}} \big|g(x)\big| = \sup_{x\in[-1/2,1/2]} \big|g(x)\big| = 1 .$$

Let $g_k(x) = \frac{g(4^{k-1}x)}{4^{k-1}}$ and $M_k = \frac{1}{4^{k-1}}$. Then $\sup_{x\in\mathbb{R}} \big|g_k(x)\big| \leqslant M_k$ and $\sum_{k=1}^{\infty} M_k < \infty$. Therefore, the Weierstrass $M$-test implies that $\sum_{k=1}^{\infty} g_k$ converges uniformly on $\mathbb{R}$. Moreover, since each $g_k$ is continuous, $\sum_{k=1}^{\infty} g_k$ is also continuous on $\mathbb{R}$.

2. We first claim that if $f$ is differentiable at $x$, then for every sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfying $a_n \leqslant x \leqslant b_n$, $b_n \neq a_n$, and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, we have

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x)\,.$$

It suffices to show the case that $a_n < x < b_n$ for all $n \in \mathbb{N}$. To see the identity above, we note that if $a_n < x < b_n$, we have

$$\Big|\frac{b_n - x}{b_n - a_n}\Big| \leqslant 1 \quad \text{and} \quad \Big|\frac{x - a_n}{b_n - a_n}\Big| \leqslant 1\,.$$

Therefore, for $a_n < x < b_n$ we have

$$\begin{aligned}
&\Big|\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'(x)\Big| \\
&\qquad = \Big|\frac{b_n - x}{b_n - a_n}\Big(\frac{f(b_n) - f(x)}{b_n - x} - f'(x)\Big) + \frac{x - a_n}{b_n - a_n}\Big(\frac{f(x) - f(a_n)}{x - a_n} - f'(x)\Big)\Big| \\
&\qquad \leqslant \Big|\frac{f(b_n) - f(x)}{b_n - x} - f'(x)\Big| + \Big|\frac{f(x) - f(a_n)}{x - a_n} - f'(x)\Big|
\end{aligned}$$

so that the Sandwich Lemma implies that $\lim\limits_{n\to\infty}\Big|\dfrac{f(b_n) - f(a_n)}{b_n - a_n} - f'(x)\Big| = 0$.

Let $\mathbb{D} = \big\{j4^{-n} \,\big|\, j, n \in \mathbb{Z}\big\}$. Suppose that $f$ is differentiable at $x \in \mathbb{R}$. Then there exists $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty} \subseteq \mathbb{D}$ such that $a_n \leqslant x < b_n$ and $b_n - a_n = \dfrac{1}{4^{n-1}}$. Then

$$f(b_n) - f(a_n) = \sum_{k=1}^{\infty} \frac{g(4^{k-1}b_n) - g(4^{k-1}a_n)}{4^{k-1}} = \sum_{k=1}^{n-1} \frac{g(4^{k-1}b_n) - g(4^{k-1}a_n)}{4^{k-1}}$$

so that

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \sum_{k=1}^{n-1} \frac{g(4^{k-1}b_n) - g(4^{k-1}a_n)}{4^{k-1}(b_n - a_n)}$$

Since $g : [4^{k-1}a_n, 4^{k-1}b_n] \to \mathbb{R}$ is "linear", we find that $\dfrac{g(4^{k-1}b_n) - g(4^{k-1}a_n)}{4^{k-1}(b_n - a_n)} = \pm 1$; thus

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \sum_{k=1}^{n-1} \pm 1$$

which does not converge by the $n$-th term test. □

**Problem 5.** Let $(M, d)$ be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\big\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \{f \in \mathscr{C}_b(A;\mathbb{R}) \,|\, f(x) > 0 \text{ for all } x \in A\}$ is open in $(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty)$.

*Proof.* 1. Let $g \in U$. By the Extreme Value Theorem, there exists $x_1, x_2 \in K$ such that

$$g(x_1) = \inf_{x\in K} g(x) \qquad \text{and} \qquad g(x_2) = \sup_{x\in K} g(x)\,.$$

Therefore, $a < \inf_{x\in K} g(x) \leqslant \sup_{x\in K} g(x) < b$. Let $r = \min\{b - \sup_{x\in K} g(x), \inf_{x\in K} g(x) - a\}$. Then $r > 0$. Moreover, if $f \in B(g,r)$ and $x \in K$, we have

$$|f(x) - g(x)| \leqslant \sup_{x\in K} |f(x) - g(x)| = \|f - g\|_\infty < r\,.$$

Therefore, if $f \in B(g,r)$, by the fact that $r \leqslant b - \sup_{x\in K} g(x)$ and $r \leqslant \inf_{x\in K} g(x) - a$, we conclude that if $x \in K$,

$$a \leqslant \inf_{x\in K} g(x) - r \leqslant g(x) - r < f(x) < g(x) + r \leqslant \sup_{x\in K} g(x) + r \leqslant b$$

which implies that $f \in U$. Therefore, $B(g,r) \subseteq U$; thus $U$ is open.

2. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $F$ such that $\{f_n\}_{n=1}^\infty$ converges uniformly to $f$ on $K$. Then $f \in \mathscr{C}(K;\mathbb{R})$. Moreover, by the fact that $a \leqslant f_n(x) \leqslant b$ for all $x \in K$ and $n \in \mathbb{N}$, we find that $a \leqslant f(x) \leqslant b$ for all $x \in K$ since $f(x) = \lim_{n\to\infty} f_n(x)$. This implies that $f \in F$; thus $F$ is closed (since it contains all the limit points).

3. Consider the case $A = (0,1)$. Then the function $f(x) = x$ belongs to $B$; however, for every $r > 0$, the function $g(x) = f(x) - \dfrac{r}{2}$ belongs to $B(f,r)$ since

$$\|f - g\|_\infty = \sup_{x\in(0,1)} |f(x) - g(x)| = \frac{r}{2} < r\,.$$

However, $g \notin B$ since if $0 < x \ll 1$, we have $g(x) < 0$. In other words, there exists no $r > 0$ such that $B(f,r) \subseteq B$; thus $B$ is not open. $\square$