Exercise Problem Sets 3

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Problem 1. Define a set $S \subseteq [0,1] \times [0,1]$ by

$$S = \Big\{ \Big(\frac{p}{m}, \frac{k}{m}\Big) \in [0,1] \times [0,1] \,\Big|\, m, p, k \in \mathbb{N}, \gcd(m,p) = 1 \text{ and } 1 \leqslant k \leqslant m-1 \Big\}.$$

Show that

$$\int_0^1 \Big(\int_0^1 \mathbf{1}_S(x,y)\,dy \Big) dx = \int_0^1 \Big(\int_0^1 \mathbf{1}_S(x,y)\,dx \Big) dy = 0$$

but $\mathbf{1}_S$ is not Riemann integrable on $[0,1] \times [0,1]$.

Proof. Note that for each $x \in [0,1]$, then $\mathbf{1}_S(x,y) \neq 0$ for only finitely many $y \in [0,1]$. Therefore, for each $x \in [0,1]$, $\mathbf{1}_S(x,\cdot)$ is Riemann integrable on $[0,1]$ and

$$\int_0^1 \mathbf{1}_S(x,y)\,dy = 0\,.$$

Similarly, for each $y \in [0,1]$, then $\mathbf{1}_S(x,y) \neq 0$ for only finitely many $x \in [0,1]$; thus for each $y \in [0,1]$, $\mathbf{1}_S(x,\cdot)$ is Riemann integrable on $[0,1]$ and

$$\int_0^1 \mathbf{1}_S(x,y)\,dx = 0\,.$$

Therefore,

$$\int_0^1 \Big(\int_0^1 \mathbf{1}_S(x,y)\,dy \Big) dx = \int_0^1 \Big(\int_0^1 \mathbf{1}_S(x,y)\,dx \Big) dy = 0\,.$$

However, for each partition $\mathcal{P}$ of $[0,1] \times [0,1]$, we have $\Delta \cap S \neq \varnothing$ for all $\Delta \in \mathcal{P}$; thus $U(\mathbf{1}_S, \mathcal{P}) = 1$ for all partition $\mathcal{P}$ of $[0,1] \times [0,1]$. Therefore,

$$\overline{\int_{A \times B}} \mathbf{1}_S(x,y)\,dy = 1$$

which, by the Fubini Theorem, implies that $\mathbf{1}_S$ is not Riemann integrable on $[0,1] \times [0,1]$. □

Problem 2. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}\,, \\ -2^{2n+1} & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n-1}, 2^{-n}),\ n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

1. Show that $\displaystyle\int_0^1 f(x,y)\,dx = 0$ for all $y \in \big[0, \frac{1}{2}\big)$.

2. Show that $\displaystyle\int_0^1 f(x,y)\,dy = 0$ for all $x \in [0,1)$.

3. Justify if the iterated (improper) integrals $\displaystyle\int_0^1 \int_0^1 f(x,y) dx dy$ and $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dy dx$ are identical.

Proof. 1. Since $f(x,0) = 0$ for all $x \in [0,1]$, we have $\int_0^1 f(x,0)\,dx = 0$. Suppose that $y \in \big(0, \frac{1}{2}\big)$. Then $y \in [2^{-n}, 2^{-n+1})$ for a unique natural number $n \geqslant 2$. In this case,

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } x \in [2^{-n}, 2^{-n+1})\,, \\ -2^{2n-1} & \text{if } x \in [2^{-n+1}, 2^{-n+2})\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\begin{aligned} \int_0^1 f(x,y)\,dx &= \int_{[2^{-n},2^{-n+1})} 2^{2n}\,dx + \int_{[2^{-n+1},2^{-n+2})} -2^{2n-1}\,dx \\ &= 2^{2n}(2^{-n+1} - 2^{-n}) - 2^{2n-1}(2^{-n+2} - 2^{-n+1}) = 0\,. \end{aligned}$$

2. Since $f(0,y)$ for all $y \in [0,1]$, we have $\int_0^1 f(0,y)\,dy = 0$. Suppose tat $x \in (0,1)$. Then $x \in [2^{-n}, 2^{-n+1})$ for a unique $n \in \mathbb{N}$. In this case,

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } y \in [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}\,, \\ -2^{2n+1} & \text{if } y \in [2^{-n-1}, 2^{-n}),\ n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\begin{aligned} \int_0^1 f(x,y)\,dy &= \int_{[2^{-n},2^{-n+1})} 2^{2n}\,dx + \int_{[2^{-n-1},2^{-n})} -2^{2n+1}\,dx \\ &= 2^{2n}(2^{-n+1} - 2^{-n}) - 2^{2n+1}(2^{-n} - 2^{-n-1}) = 0\,. \end{aligned}$$

3. By 2, we immediately conclude that

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = 0\,.$$

On the other hand, note that if $y \in \big[\frac{1}{2}, 1\big)$, then $f(x,y) = \begin{cases} 4 & \text{if } x \in \big[\frac{1}{2}, 1\big)\,, \\ 0 & \text{otherwise}\,, \end{cases}$ so that

$$\int_0^1 f(x,y)\,dx = \int_{\frac{1}{2}}^1 4\,dx = 2\,.$$

Therefore,

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^{\frac{1}{2}} \int_0^1 f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^1 \int_0^1 f(x,y)\,dx\,dy = \int_{\frac{1}{2}}^1 2\,dy = 1$$

which shows that $\int_0^1 \int_0^1 f(x,y) dx dy \neq \int_0^1 \int_0^1 f(x,y)\,dy dx$ for this particular f. □

Problem 3. Suppose that $f : (0,b] \to \mathbb{R}$ is continuous, positive, integrable on $(0,b]$, and that $f(x)$ increases monotonically to ∞ as x approaches 0 from the right. Show that $\lim\limits_{x \to 0^+} x f(x) = 0$.

Proof. Let $\limsup_{x \to 0^+} xf(x) = L$. Then $L \geqslant 0$, and there exists a sequence $\{x_k\}_{k=1}^{\infty} \subseteq (0, b]$ such that $\lim_{k \to \infty} x_k f(x_k) = L$. W.L.O.G. we can assume that the sequence $x_{k+1} < \frac{x_k}{2}$ for all $k \in \mathbb{N}$. If $L > 0$, then there exists $N > 0$ such that

$$x_k f(x_k) > \frac{L}{2} \qquad \forall\, k \geqslant N$$

so that $f(x_k) > \frac{L}{2x_k}$ whenever $k \geqslant N$. Therefore, by the monotonicity of f we find that

$$f(x) > \frac{L}{2x_k} \qquad \forall\, x \in [x_{k+1}, x_k] \text{ and } k \geqslant N\,.$$

Therefore,

$$\int_{(0,b]} f(x)\,dx \geqslant \sum_{k=N}^{\infty} (x_k - x_{k+1}) \frac{L}{2x_k} \geqslant \sum_{k=N}^{\infty} \frac{x_k}{2} \cdot \frac{L}{2x_k} = \sum_{k=N}^{\infty} \frac{L}{4} = \infty\,,$$

a contradiction to that f is integrable on $(0, b]$. □

Problem 4. Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be Riemann measurable sets, and $f : A \times B \to \mathbb{R}$ be non-negative, uniformly continuous and integrable on $A \times B$. Define $F(x) = \int_B f(x, y)\,dy$.

1. Show that if B is bounded, then $F : A \to \mathbb{R}$ is continuous. How about if B is not bounded?

2. Let f have the additional property that for each $\varepsilon > 0$, there exists $N > 0$ such that

$$\Big| \int_{B \cap B(0,k)} (f \wedge k)(x, y)\,dy - \int_B f(x, y)\,dy \Big| < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

Show that F is continuous on A. In particular, show that if $f(x, y) \leqslant g(y)$ for all $(x, y) \in A \times B$, and g is integrable on B, then F is continuous.

Proof. 1. If B is bounded, then B has volume. Let $\varepsilon > 0$ be given. By the uniform continuity of f, there exists $\delta > 0$ such that

$$\big|f(x_1, y_1) - f(x_2, y_2)\big| < \frac{\varepsilon}{\nu(B) + 1} \qquad \forall\, \big|(x_1, y_1) - (x_2, y_2)\big| < \delta \text{ and } x_1, x_2 \in A, y_1, y_2 \in B\,.$$

Therefore, if $|x_1 - x_2| < \delta$ and $x_1, x_2 \in A$,

$$\begin{aligned} \big|F(x_1) - F(x_2)\big| &= \Big| \int_B \big[f(x_1, y) - f(x_2, y)\big]\,dy \Big| \leqslant \int_B \big|f(x_1, y) - f(x_2, y)\big|\,dy \\ &\leqslant \int_B \frac{\varepsilon}{\nu(B) + 1}\,dx \leqslant \frac{\varepsilon \nu(B)}{\nu(B) + 1} < \varepsilon\,. \end{aligned}$$

This implies that F is uniformly continuous on A.

If B is unbounded, then the argument above does not apply. In fact, consider the case

$$f(x, y) = \frac{\sqrt{x}}{1 + x^2 y^2}\,, \quad A = [0, 1] \quad \text{and} \quad B = \mathbb{R}\,.$$

Then f is non-negative and uniformly continuous on $A \times B$ (why?). Note that $F(0) = 0$ while if $x > 0$,

$$F(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{-\infty}^{\infty} \frac{\sqrt{x}}{1+x^2y^2}\,dy = \frac{\sqrt{x}}{x} \arctan(xy)\Big|_{y=-\infty}^{y=\infty} = \frac{\pi}{\sqrt{x}}.$$

Therefore, the Tonelli Theorem implies that

$$\int_{A\times B} f(x,y)\,d(x,y) = \int_A \Big(\int_B f(x,y)\,dy\Big)\,dx = \int_0^1 \frac{\pi}{\sqrt{x}}\,dx = 2\pi < \infty$$

which shows that f is integrable on $A \times B$. However, F is not continuous at $x = 0$.

2. Let $\varepsilon > 0$ be given. Since f has the property mentioned above, there exists $N > 0$ such that

$$\Big|\int_{B\cap B(0,k)} (f \wedge k)(x,y)\,dy - \int_B f(x,y)\,dy\Big| < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

By the uniform continuity of f on $A \times B$, there exists $\delta > 0$ such that

$$\big|f(x_1,y_1) - f(x_2,y_2)\big| < \frac{\varepsilon}{3\nu(B(0,N))} \qquad \forall\,\big|(x_1,y_1)-(x_2,y_2)\big| < \delta \text{ and } x_1,x_2 \in A, y_1,y_2 \in B\,.$$

Suppose that $|x_1 - x_2| < \delta$, $x_1, x_2 \in A$ and $y \in B$.

(a) If $f(x_1,y)$ and $f(x_2,y)$ are both not greater than N, then

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| = \big|f(x_1,y) - f(x_2,y)\big| < \frac{\varepsilon}{3\nu(B(0,N))}\,.$$

(b) If $f(x_1,y)$ and $f(x_2,y)$ are both greater than N, then

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| = |N - N| = 0\,.$$

(c) If one and only one of $f(x_1,y)$ and $f(x_2,y)$ is greater than N, then

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| < \big|f(x_1,y) - f(x_2,y)\big| < \frac{\varepsilon}{3\nu(B(0,N))}\,.$$

Case (a), (b) and (c) show that

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| < \frac{\varepsilon}{3\nu(B(0,N))} \qquad \forall\,|x_1 - x_2| < \delta, x_1,x_2 \in A \text{ and } y \in B\,.$$

Therefore, if $x_1, x_2 \in A$ and $|x_1 - x_2| < \delta$,

$$\begin{aligned}
\big|F(x_1) - F(x_2)\big| &\leqslant \Big|\int_{B\cap B(0,N)} (f\wedge N)(x_1,y)\,dy - \int_B f(x_1,y)\,dy\Big| \\
&\quad + \Big|\int_{B\cap B(0,N)} (f\wedge N)(x_2,y)\,dy - \int_B f(x_2,y)\,dy\Big| \\
&\quad + \Big|\int_{B\cap B(0,N)} (f\wedge N)(x_1,y)\,dy - \int_{B\cap B(0,N)} (f\wedge N)(x_2,y)\,dy\Big| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int_{B\cap B(0,N)} \big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big|\,dy \leqslant \varepsilon\,.
\end{aligned}$$

This implies that F is uniformly continuous on A.

Now suppose that $f(x,y) \leqslant g(y)$ for all $(x,y) \in A \times B$, and g is integrable on B. Then

$$\lim_{k\to\infty} \int_{B\cap B(0,k)} (g \wedge k)(y)\, dy = \int_B g(y)\, dy\,;$$

thus there exists $N > 0$ such that

$$\Big| \int_{B\cap B(0,k)} (g \wedge k)(y)\, dy - \int_B g(y)\, dy \Big| < \varepsilon \qquad \text{whenever} \quad k \geqslant N\,.$$

Therefore, for all $k \geqslant N$ and $x \in A$,

$$\begin{aligned}
&\Big| \int_{B\cap B(0,k)} (f \wedge k)(x,y)\, dy - \int_B f(x,y)\, dy \Big| \\
&\quad \leqslant \Big| \int_{B\cap B(0,k)} (f \wedge k)(x,y)\, dy - \int_{B\cap B(0,k)} f(x,y)\, dy \Big| + \int_{B\cap B(0,k)^\complement} f(x,y)\, dy \\
&\quad \leqslant \int_{B\cap B(0,k)} \big|(f \wedge k)(x,y) - f(x,y)\big|\, dy + \int_{B\cap B(0,k)^\complement} g(y)\, dy \\
&\quad \leqslant \int_{\{y\in B\cap B(0,k)\,|\, f(x,y)>k\}} \big[f(x,y) - k\big]\, dy + \int_{B\cap B(0,k)^\complement} g(y)\, dy \\
&\quad \leqslant \int_{\{y\in B\cap B(0,k)\,|\, g(y)>k\}} \big[g(y) - k\big]\, dy + \int_{B\cap B(0,k)^\complement} g(y)\, dy \\
&\quad \leqslant \int_{B\cap B(0,k)} \big[g(y) - (g\wedge k)(y)\big]\, dy + \int_{B\cap B(0,k)^\complement} g(y)\, dy \\
&\quad = \int_B g(y)\, dy - \int_{B\cap B(0,k)} (g \wedge k)(y)\, dy < \varepsilon\,.
\end{aligned}$$

This shows that f satisfies the condition mentioned in 2; thus F is continuous on A. ▫

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable Riemann measurable function, and $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \int_{\mathbb{R}} f(y) \cos(x-y)\, dy$$

(which exists for all $x \in \mathbb{R}$ since f is integrable). Show that F is differentiable on $\mathbb{R}$ and

$$F'(x) = \int_{\mathbb{R}} f(y) \frac{\partial}{\partial x} \cos(x-y)\, dx = -\int_{\mathbb{R}} f(y) \sin(x-y)\, dx\,.$$

Hint: Apply the Dominated Convergence Theorem.

Proof. Let $x \in \mathbb{R}$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence with limit 0. Define

$$g_k(y) = f(y) \frac{\cos(x+h_k-y) - \cos(x-y)}{h_k}\,.$$

Then for all $y \in \mathbb{R}$, $\lim\limits_{k\to\infty} g_k(y) = f(y)\dfrac{\partial}{\partial x}(cos(x-y)) = -f(y)\sin(x-y)$.

Since $\Big|\dfrac{d}{dx}\cos x\Big| \leqslant 1$, the mean value theorem implies that

$$\big|\cos(x+h_k-y) - \cos(x-y)\big| \leqslant |h_k|\,.$$

Therefore,

$$\big|g_k(y)\big| \leqslant \big|f(y)\big| \qquad \forall\, x \in \mathbb{R}\,.$$

Since f is integrable on $\mathbb{R}$, $|f|$ is integrable on $\mathbb{R}$; thus the Dominated Convergence Theorem implies that

$$\lim_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k} = \lim_{k\to\infty} \int_{\mathbb{R}} g_k(y)\,dx = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dx\,.$$

The equality above shows that for each non-zero sequence $\{h_k\}_{k=1}^\infty$ with limit 0, the limit

$$\lim_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k} = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dx$$

exists. By the definition of the limit of functions,

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dx\,. \qquad \square$$

Problem 6. Let $f:\mathbb{R}\to\mathbb{R}$ be an integrable Riemann measurable function, and $F:\mathbb{R}\to\mathbb{R}$ be defined by

$$F(x) = \int_{\mathbb{R}} f(y)\cos(xy)\,dy$$

(which exists for all $x \in \mathbb{R}$ since f is integrable). Show that if the function $g(x) = xf(x)$ is integrable, then F is differentiable on $\mathbb{R}$ and

$$F'(y) = \int_{\mathbb{R}} f(x)\frac{\partial}{\partial y}\cos(xy)\,dx = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx\,.$$

Hint: Apply the Dominated Convergence Theorem.

Proof. Let $y \in \mathbb{R}$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence with limit 0. Define

$$g_k(x) = f(x)\frac{\cos(x(y+h_k))-\cos(xy)}{h_k}\,.$$

Then for all $x \in \mathbb{R}$, $\lim\limits_{k\to\infty} g_k(x) = f(x)\dfrac{\partial}{\partial y}(cos(xy)) = -xf(x)\sin(xy)$.

Since $\left|\dfrac{d}{dx}\cos x\right| \leqslant 1$, the mean value theorem implies that

$$\big|\cos(x(y+h_k))-\cos(xy)\big| \leqslant |xh_k|\,.$$

Therefore,

$$\big|g_k(x)\big| \leqslant \big|xf(x)\big| = \big|g(x)\big| \qquad \forall\, x \in \mathbb{R}\,.$$

Since g is integrable on $\mathbb{R}$, $|g|$ is integrable on $\mathbb{R}$; thus the Dominated Convergence Theorem implies that

$$\lim_{k\to\infty} \frac{F(y+h_k)-F(y)}{h_k} = \lim_{k\to\infty} \int_{\mathbb{R}} h_k(x)\,dx = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx\,.$$

The equality above shows that for each non-zero sequence $\{h_k\}_{k=1}^\infty$ with limit 0, the limit

$$\lim_{k\to\infty} \frac{F(y+h_k)-F(y)}{h_k} = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx$$

exists. By the definition of the limit of functions,

$$\lim_{h\to 0} \frac{F(y+h)-F(y)}{h} = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx\,. \qquad \square$$

Problem 7. Let $f(x,y) = \begin{cases} \dfrac{e^{-xy}\sin y}{y} & \text{if } y \neq 0\,, \\ 1 & \text{if } y = 0\,. \end{cases}$.

1. Show that $f_x(x,y)$ is continuous everywhere, and show that $f(x,\cdot)$ is integrable on $[0,\infty)$ for all $x > 0$.
2. Define $F(x) = \displaystyle\int_0^\infty f(x,y)\,dy$ for $x > 0$. Show that $F'(x) = -\dfrac{1}{x^2+1}$.
3. Show that $F(x) = \dfrac{\pi}{2} - \arctan x$ if $x > 0$, and conclude that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}\,.$$

Proof. 1. Note that if $y \neq 0$, $f_x(x,y) = e^{-xy}\sin y$ while $f_x(x,0) = 0$. Clearly f_x is continuous on $\mathbb{R}^2$ except perhaps on the x-axis. On the other hand, since $\displaystyle\lim_{(x,y)\to(a,0)} f(x,y) = 0$, we conclude that f_x is also continuous on the x-axis. Therefore, f_x is continuous everywhere.

Let $x > 0$ be given. Then $\big|f(x,y)\big| \leqslant e^{-xy}$. Since the right-hand side function, for given $x > 0$, is integrable on $[0,\infty)$, the comparison test implies that $f(x,\cdot)$ is integrable on $[0,\infty)$.

2. Let $x > 0$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence with limit 0. W.L.O.G., we can assume that $|h_k| < \dfrac{x}{2}$ since $x > 0$. Define

$$g_k(y) = \begin{cases} \dfrac{e^{-yh_k}-1}{h_k} e^{-xy} \dfrac{\sin y}{y} & \text{if } y \neq 0\,, \\ 0 & \text{if } y = 0\,. \end{cases}$$

The mean value theorem implies that $\left|\dfrac{e^{-yh_k}-1}{h_k}\right| \leqslant e^{\frac{xy}{2}}|y|$; thus

$$\big|g_k(y)\big| \leqslant e^{-\frac{xy}{2}} \qquad \forall\, y \geqslant 0\,.$$

Since the right-hand side function, for given $x > 0$, is integrable on $[0,\infty)$, the Dominated Convergence Theorem implies that

$$\begin{aligned} \lim_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k} &= \lim_{k\to\infty} \int_0^\infty \frac{f(x+h_k,y)-f(x,y)}{h_k}\,dy = \lim_{k\to\infty}\int_0^\infty g_k(y)\,dy \\ &= \int_0^\infty \lim_{k\to\infty} g_k(y)\,dy = -\int_0^\infty e^{-xy}\sin y\,dy \end{aligned}$$

Integrating by parts, by the fact $x > 0$ we find that

$$\begin{aligned} \int_0^\infty e^{-xy}\sin y\,dy &= -e^{-xy}\cos y\Big|_{y=0}^{y=\infty} - x\int_0^\infty e^{-xy}\cos y\,dy \\ &= 1 - x\Big[e^{-xy}\sin y\Big|_{y=0}^{y=\infty} + x\int_0^\infty e^{-xy}\sin y\,dy\Big] \\ &= 1 - x^2\int_0^\infty e^{-xy}\sin y\,dy\,; \end{aligned}$$

thus we conclude that

$$\lim_{k\to\infty}\frac{F(x+h_k)-F(x)}{h_k} = -\frac{1}{1+x^2} \text{ for all } x>0 \text{ and non-zero sequence } \{h_k\}_{k=1}^{\infty} \text{ with limit } 0.$$

Therefore, for $x > 0$ the limit $\displaystyle\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$ exists (so that F is differentiable on $(0,\infty)$) and

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = -\frac{1}{1+x^2} \qquad \forall\, x>0\,.$$

3. By the (generalized version of) Fundamental Theorem of Calculus, for $a, b > 0$ we have

$$F(b)-F(a) = \int_a^b F'(x)\,dx = -\int_a^b \frac{1}{1+x^2}\,dx = \arctan x\Big|_{x=a}^{x=b} = \arctan a - \arctan b\,.$$

Note that for $a > 0$ we have

$$|F(a)| \leqslant \int_0^\infty e^{-ay}\,dy = \frac{e^{-ay}}{-a}\Big|_{y=0}^{y=\infty} = \frac{1}{a}\,;$$

thus $\displaystyle\lim_{a\to\infty} F(a) = 0$ by the Sandwich lemma. Therefore, for $x > 0$,

$$F(x) = F(x) - \lim_{a\to\infty} F(a) = \lim_{a\to\infty}\big[F(x)-F(a)\big] = \lim_{a\to\infty}\big(\arctan a - \arctan x\big) = \frac{\pi}{2} - \arctan x\,.$$

Finally, we show that $F(0) = \displaystyle\lim_{x\to 0^+} F(x)$. Let $\varepsilon > 0$ be given. Since

$$\frac{\partial}{\partial y}\Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big) = (e^{-xy}-1)\sin y\,,$$

integrating by parts shows that for all $n > 0$,

$$\begin{aligned}\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy &= \frac{1}{y}\Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big)\Big|_{y=n}^{y=\infty} \\ &\quad + \int_n^\infty \Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big)\frac{1}{y^2}\,dy\,.\end{aligned}$$

By the fact that

$$\Big|\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big| \leqslant \frac{x+1}{x^2+1} + 1 \leqslant \frac{5}{2} < 3\,,$$

we have

$$\Big|\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \leqslant \int_n^\infty \frac{3}{y^2}\,dy + \frac{3}{n} = \frac{6}{n}\,.$$

Therefore, for all $n > 0$,

$$\begin{aligned}\big|F(x)-F(0)\big| &= \Big|\int_0^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \\ &\leqslant \Big|\int_0^n (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| + \Big|\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \\ &\leqslant \int_0^n (1-e^{-xy})\,dy + \frac{6}{n} \\ &= \Big(y + \frac{e^{-xy}}{x}\Big)\Big|_{y=0}^{y=n} + \frac{6}{n} = n + \frac{e^{-nx}-1}{x} + \frac{6}{n}\end{aligned}$$

so that

$$\limsup_{x\to 0^+} \big|F(x) - F(0)\big| \leqslant \frac{6}{n} \qquad \forall\, n > 0\,.$$

Since $n > 0$ is given arbitrarily, we conclude that $\limsup\limits_{x\to 0^+} \big|F(x) - F(0)\big| = 0$ which shows that $\lim\limits_{x\to 0^+} F(x) = F(0)$. As a consequence,

$$\int_0^\infty \frac{\sin x}{x}\, dx = F(0) = \lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \big(\frac{\pi}{2} - \arctan x\big) = \frac{\pi}{2}\,. \qquad \square$$