Exercise Problem Sets 2

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Problem 1. Let $A = \bigcup\limits_{k=1}^{\infty} B\big(\frac{1}{k}, \frac{1}{2^k}\big) = \bigcup\limits_{k=1}^{\infty} \big(\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big)$ be a subset of $\mathbb{R}$. Does A have volume?

Proof. We first show that $\bar{A} = \{0\} \cup \bigcup\limits_{k=N+1}^{\infty} \big[\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big]$.

1. Clearly $\bigcup\limits_{k=N+1}^{\infty} \big[\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big] \subseteq \bar{A}$. In fact, we have $\bigcup\limits_{\alpha \in I} \overline{B_\alpha} \subseteq \text{cl}\Big(\bigcup\limits_{\alpha \in I} B_\alpha\Big)$: if $x \in \bigcup\limits_{\alpha \in I} \overline{B_\alpha}$, then $x \in \overline{B_\alpha}$ for some $\alpha \in I$ which implies that there exists $\alpha \in I$ and $\{x_\ell\}_{\ell=1}^{\infty} \in B_\alpha \subseteq \bigcup\limits_{\alpha \in I} B_\alpha$ such that $x_\ell \to x$ as $\ell \to \infty$. Therefore, $x \in \text{cl}\Big(\bigcup\limits_{\alpha \in I} B_\alpha\Big)$.

2. Suppose that $x \in \bar{A}$. Then there exists $\{x_\ell\}_{\ell=1}^{\infty} \subseteq A$ such that $x_\ell \to x$ as $\ell \to \infty$. Since every element in A is positive, we conclude that $x \geqslant 0$.

   (a) **the case** $x = 0$: Since $\{x_\ell\}_{\ell=1}^{\infty}$ defined by $x_\ell = \frac{1}{\ell}$ is a sequence in A, we conclude that $0 \in \bar{A}$ since $\lim\limits_{\ell \to \infty} x_\ell = 0$.

   (b) **the case** $x > 0$: By the definition of the limit of sequences, there exists $N > 0$ such that $x_\ell \in \big(\frac{x}{2}, \frac{3x}{2}\big)$ for all $\ell \geqslant N$. Since $\lim\limits_{k \to \infty} \frac{1}{k} + \frac{1}{2^k} = 0$, there exists $M > 0$ such that $\frac{1}{k} + \frac{1}{2^k} < \frac{x}{2}$ for all $k \geqslant M$. Therefore,

   $$A \cap \big(\frac{x}{2}, \frac{3x}{2}\big) = \bigcup_{k=1}^{M-1} \big(\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big);$$

   thus there exists $1 \leqslant j \leqslant M-1$ such that

   $$\#\big\{\ell \in \mathbb{N} \,\big|\, x_\ell \in \big(\frac{1}{j} - \frac{1}{2^j}, \frac{1}{j} + \frac{1}{2^j}\big) = \infty\,.$$

   Let $\{x_{\ell_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_\ell\}_{\ell=1}^{\infty}$ satisfying that $\{x_{\ell_k}\}_{k=1}^{\infty} \subseteq \big(\frac{1}{j} - \frac{1}{2^j}, \frac{1}{j} + \frac{1}{2^j}\big)$, we conclude that $x \in \big[\frac{1}{j} - \frac{1}{2^j}, \frac{1}{j} + \frac{1}{2^j}\big]$ since $\lim\limits_{k \to \infty} x_{\ell_k} = x$.

Having shown that $\bar{A} = \{0\} \cup \bigcup\limits_{k=N+1}^{\infty} \big[\frac{1}{k} - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}\big]$, we conclude that

$$\partial A = \bar{A} \backslash \mathring{A} = \bar{A} \backslash A \subseteq \{0\} \cup \Big\{\frac{1}{k} - \frac{1}{2^k} \,\Big|\, k \in \mathbb{N}\Big\} \cup \Big\{\frac{1}{k} + \frac{1}{2^k} \,\Big|\, k \in \mathbb{N}\Big\};$$

thus ∂A is a countable set which has measure zero. This implies that A has volume. □

Problem 2. Complete the following.

1. Let $A \subseteq [a, b]$ be a set of measure zero (in $\mathbb{R}$). Show that $[a, b] \backslash A$ does not have measure zero (in $\mathbb{R}$).

2. Show that the Cantor set (defined in Problem 9 of Exercise 7 from the previous semester) has volume zero.

Proof. 1. Suppose the contrary that $[a,b]\backslash A$ has measure zero. By the fact that countable union of measure zero sets has measure zero, we conclude that

$$[a,b] = A \cup ([a,b]\backslash A)$$

has measure zero, a contradiction.

2. Let E_n be the set defined in Problem 9 of Exercise 7. Then E_n is the union of finite intervals whose volumes sum to $\frac{2^n}{3^n}$. Therefore, for each $\varepsilon > 0$ there exist finite rectangles $S_1, S_2, \cdots, S_N$ whose disjoint union is E_N and $\sum\limits_{k=1}^{N} \nu(S_k) = \frac{2^N}{3^N} < \varepsilon$. This shows that the Cantor set has volume zero. □

Problem 3. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. Show that if f is Riemann integrable on A, then $|f|$ is also Riemann integrable on A.

Proof. **Method 1**: Since f is Riemnn integrable on A, the Lebesgue Theorem implies that the collection of discontinuities of $\overline{f}^A$ has measure zero. Note that if $\overline{f}^A$ is continuous at $a \in A$, then $\overline{|f|}^A$ is also continuous at a since $\overline{|f|}^A = |\overline{f}^A|$. Therefore, the collection of discontinuities of $\overline{|f|}^A$ is a subset of a measure zero set, the collection of discontinuities of $\overline{f}^A$; thus the collection of discontinuities of $\overline{|f|}^A$ has measure zero. The Lebesgue Theorem then shows that $|f|$ is Riemann integrable on A.

Method 2: Let $\varepsilon > 0$ be given. Since f is Riemann integrable on A, by Riemann's condition there exists a partition $\mathcal{P}$ of A such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon\,.$$

Note that for each $\Delta \in P$,

$$\sup_{x\in\Delta} |\overline{f}^A(x)| - \inf_{x\in\Delta} |\overline{f}^A(x)| \leqslant \sup_{x\in\Delta} \overline{f}^A(x) - \inf_{x\in\Delta} \overline{f}^A(x)\,;$$

thus

$$\begin{aligned} U(|f|,\mathcal{P}) - L(|f|,\mathcal{P}) &= \sum_{\Delta\in\mathcal{P}} \Big(\sup_{x\in\Delta} |\overline{f}^A(x)| - \inf_{x\in\Delta} |\overline{f}^A(x)|\Big)\nu(\Delta) \\ &\leqslant \sum_{\Delta\in\mathcal{P}} \Big(\sup_{x\in\Delta} \overline{f}^A(x) - \inf_{x\in\Delta} \overline{f}^A(x)\Big)\nu(\Delta) = U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon\,. \end{aligned}$$

Therefore, by Riemann's condition we conclude that $|f|$ is Riemann integrable on A. □

Problem 4. Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable, and the set $\big\{x \in [a,b] \,\big|\, f(x) \neq 0\big\}$ has measure zero. Show that $\int_a^b f(x)\,dx = 0$.

Proof. First we note that for each $[c,d] \subseteq [a,b]$, then there exists $x \in [c,d]$ such that $f(x) = 0$ for otherwise $f(x) \neq 0$ for all $x \in [c,d]$ so that

$$[c,d] \subseteq \{x \in [a,b] \,\big|\, f(x) \neq 0\}$$

and this implies that $[c,d]$ is a set of measure zero, a contradiction. Therefore, $L(|f|, \mathcal{P}) = 0$ for all partitions $\mathcal{P}$ of $[a,b]$ which shows that $\underline{\int_a^b} \big|f(x)\big|\,dx = 0$. Since f is Riemann integrable on $[a,b]$, $|f|$ is also Riemann integrable on $[a,b]$ so that we have $\int_a^b \big|f(x)\big|\,dx = 0$. The desired conclusion then follows from the fact that

$$\Big|\int_a^b f(x)\,dx\Big| \leqslant \int_a^b \big|f(x)\big|\,dx\,. \qquad \square$$

Remark: Similar argument can be used to show the following:

1. Let $A \subseteq \mathbb{R}^n$ be a bounded set and $f : A \to \mathbb{R}$ be Riemann integrable on A. If the set $\{x \in A \,\big|\, f(x) \neq 0\}$ has measure zero, then $\int_A f(x)\,dx = 0$.
2. Let $A \subseteq \mathbb{R}^n$ be a bounded set and $f : A \to \mathbb{R}$ be a function. If the set $\{x \in A \,\big|\, f(x) \neq 0\}$ has measure zero, then $\underline{\int_A} f(x)\,dx \leqslant 0$.

Problem 5. Prove the following statements.

1. The function $f(x) = \sin\dfrac{1}{x}$ is Riemann integrable on $(0,1)$.
2. Let $f : (0,1] \to \mathbb{R}$ be given by

   $$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \in \mathbb{Q},\ (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Then f is Riemann integrable on $(0,1]$. Find $\int_{(0,1]} f(x)dx$ as well.
3. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ is Riemann integrable. Then f^k (f 的 k 次方) is integrable for all $k \in \mathbb{N}$.

Proof. 1. Note that $(0,1)$ has volume, f is bounded on $(0,1)$ and f is continuous on $(0,1)$. Therefore, the Lebesgue Theorem (or its corollary) implies that f is Riemann integrable on $(0,1)$.

2. In Calculus we have shown that f is continuous on $\mathbb{Q}^\complement \cap (0,1]$ so that the collection of discontinuities of $\overline{f}^{(0,1]}$ is $\mathbb{Q} \cap (0,1]$. Since $\mathbb{Q} \cap (0,1]$ is countable, we find that the collection of discontinuities of $\overline{f}^{(0,1]}$ has measure zero. Therefore, f is Riemann integrable on $(0,1]$.

   Let $\mathcal{P}$ be a partition of $(0,1]$. Then $L(f, \mathcal{P}) = 0$ since

   $$\inf_{x \in \Delta} \overline{f}^{(0,1]}(x) = 0 \qquad \forall\, \Delta \in \mathcal{P}\,.$$

   Therefore, $\underline{\int_A} f(x)\,dx = 0$. The Riemann integrability of f then shows that $\int_{(0,1]} f(x)\,dx = 0$.

3. First we note that the fact that f is Riemann integrable on A implies that f is bounded on A. Therefore, f^k is bounded on A. Moreover, the Lebesgue Theorem implies that the collection D of discontinuities of $\overline{f}^A$ has measure zero. Since $\overline{f^k}^A = \big(\overline{f}^A\big)^k$, we find that the collection of discontinuities of $\overline{f^k}^A$ is exactly D; thus has measure zero. The Lebesgue Theorem then implies that f^k is Riemann integrable on A. $\square$

Problem 6. Find an example that

$$\underline{\int_A} f(x)\,dx + \underline{\int_A} g(x)\,dx < \underline{\int_A} (f+g)(x)\,dx < \overline{\int_A} (f+g)(x)\,dx < \overline{\int_A} f(x)\,dx + \overline{\int_A} g(x)\,dx\,.$$

Solution. Let $f, g : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 0 & \text{if } x \in \mathbb{Q}^{\complement} \cap [0,1]\,, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 2 & \text{if } x \in \mathbb{Q}^{\complement} \cap [0,1]\,, \end{cases}$$

Then

$$(f+g)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 2 & \text{if } x \in \mathbb{Q}^{\complement} \cap [0,1]\,, \end{cases}$$

so that we have $\underline{\int_{[0,1]}} f(x)\,dx = \underline{\int_{[0,1]}} g(x)\,dx = 0$, $\overline{\int_{[0,1]}} f(x)\,dx = \underline{\int_{[0,1]}} (f+g)(x)\,dx = 1$, and $\overline{\int_{[0,1]}} g(x)\,dx = \overline{\int_{[0,1]}} (f+g)(x)\,dx = 2$. $\square$