Exercise Problem Sets 11

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Problem 1. Let $\{T_k\}_{k=1}^{\infty} \subseteq \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ be a sequence of bounded linear maps from $\mathbb{R}^n \to \mathbb{R}^m$. Prove that the following three statements are equivalent:

1. there exists a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $\{T_k \boldsymbol{x}\}_{k=1}^{\infty}$ converges to $T\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$;

2. $\lim\limits_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$;

3. there exists a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that for every compact $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ there exists $N > 0$ such that

$$\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \varepsilon \qquad \text{whenever} \quad \boldsymbol{x} \in K \ \text{ and } \ k \geqslant N\,.$$

Proof. "1 ⇒ 3" Let K be a compact set in $\mathbb{R}^n$, and $\varepsilon > 0$ be given. Then there exists $R > 0$ such that $K \subseteq B[0, R]$. By assumption, for each $1 \leqslant i \leqslant n$, there exist $N_i > 0$ such that

$$\|T_k\mathbf{e}_i - T\mathbf{e}_i\|_{\mathbb{R}^m} < \frac{\varepsilon}{Rn} \qquad \text{whenever} \quad k \geqslant N_i\,.$$

For $\boldsymbol{x} \in \mathbb{R}^n$, write $\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n$. Then if $\boldsymbol{x} \in K$, $\left|x^{(i)}\right| \leqslant R$ for all $1 \leqslant i \leqslant n$. Therefore, if $\boldsymbol{x} \in K$ and $k \geqslant N \equiv \max\{N_1, \cdots, N_n\}$,

$$\begin{aligned}\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} &= \Big\|T_k\Big(\sum_{i=1}^{n} x^{(i)}\mathbf{e}_i\Big) - T\Big(\sum_{i=1}^{n} x^{(i)}\mathbf{e}_i\Big)\Big\|_{\mathbb{R}^m} = \Big\|\sum_{i=1}^{n} x^{(i)}\big(T_k\mathbf{e}_i - T\mathbf{e}_i\big)\Big\|_{\mathbb{R}^m} \\ &\leqslant \sum_{i=1}^{n}\left|x^{(i)}\right|\|T_k\mathbf{e}_i - T\mathbf{e}_i\|_{\mathbb{R}^m} < \sum_{i=1}^{n} R\frac{\varepsilon}{Rn} = \varepsilon\,.\end{aligned}$$

"3 ⇒ 2" Let $K = B[0, 1]$ (which is compact), and $\varepsilon > 0$ be given. By assumption there exists $N > 0$ such that

$$\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \frac{\varepsilon}{3} \qquad \text{whenever} \quad \boldsymbol{x} \in B[0,1] \ \text{ and } \ k \geqslant N\,.$$

If $k, \ell \geqslant N$ and $\boldsymbol{x} \in B[0, 1]$,

$$\|T_k\boldsymbol{x} - T_\ell x\|_{\mathbb{R}^m} \leqslant \|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} + \|T_\ell\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \frac{2\varepsilon}{3}$$

which shows that

$$\|T_k - T_\ell\| = \sup_{\boldsymbol{x}\in B[0,1]} \|T_k\boldsymbol{x} - T_\ell x\|_{\mathbb{R}^m} \leqslant \frac{2\varepsilon}{3} < \varepsilon \qquad \forall\, k, \ell \geqslant N\,.$$

Therefore, $\lim\limits_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$.

"2 ⇒ 1" By assumption, for each $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$\|T_k\boldsymbol{x} - T_\ell\boldsymbol{x}\|_{\mathbb{R}^m} = \|(T_k - T_\ell)\boldsymbol{x}\|_{\mathbb{R}^m} \leqslant \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|\boldsymbol{x}\|_{\mathbb{R}^n} \to 0 \quad \text{as} \quad k,\ell \to \infty\,.$$

Therefore, for each $\boldsymbol{x} \in \mathbb{R}^n$ the sequence $\{T_k\boldsymbol{x}\}_{k=1}^\infty$ is Cauchy in $\mathbb{R}^m$; thus convergent. So we establish a map $x \mapsto \lim\limits_{k\to\infty} T_k\boldsymbol{x}$ which is denoted by T. In other words, $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\boldsymbol{x}) = \lim\limits_{k\to\infty} T_k\boldsymbol{x}$.

If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

$$T(c\boldsymbol{x}_1 + \boldsymbol{x}_2) = \lim_{k\to\infty} T_k(c\boldsymbol{x}_1 + \boldsymbol{x}_2) = \lim_{k\to\infty} \big(cT_k\boldsymbol{x}_1 + T_k\boldsymbol{x}_2\big) = cT(\boldsymbol{x}_1) + T(\boldsymbol{x}_2)\,.$$

Therefore, $T \in \mathscr{L}(\mathbb{R}^n,\mathbb{R}^m)$. Since $\dim(\mathbb{R}^n) = n < \infty$, we conclude that $T \in \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$. □

Problem 2. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x,y,z) = (x^2+y^2)\mathbf{i}+xyz\,\mathbf{j}$. Show that f is differentiable on $\mathbb{R}^3$ and find $(Df)(a,b,c)$.

Proof. For $(h,k,\ell) \in \mathbb{R}^3$,

$$\begin{aligned} &f(a+h,b+k,c+\ell) - f(a,b,c) \\ &\qquad = \big((a+h)^2 + (b+k)^2\big)\mathbf{i} + (a+h)(b+k)(c+\ell)\,\mathbf{j} - (a^2+b^2)\mathbf{i} - abc\,\mathbf{j} \\ &\qquad = (2ah + 2bk + h^2 + k^2)\mathbf{i} + (ab\ell + bch + ack + ak\ell + bh\ell + chk + hk\ell)\,\mathbf{j}\,; \end{aligned}$$

thus we expect that

$$(Df)(a,b,c)(h,k,\ell) = (2ah + 2bk)\mathbf{i} + (ab\ell + bch + ack)\,\mathbf{j}\,. \tag{$\diamond$}$$

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(h,k,\ell) = (2ah+2bk)\mathbf{i}+(ab\ell+bch+ack)\,\mathbf{j}$. Clearly $L \in \mathscr{B}(\mathbb{R}^3,\mathbb{R}^2)$. Moreover, for $(h,k,\ell) \neq (0,0,0)$,

$$\begin{aligned} &\frac{\big\|f(a+h,b+k,c+\ell) - f(a,b,c) - L(h,k,\ell)\big\|}{\sqrt{h^2+k^2+\ell^2}} \\ &\quad = \frac{\big\|(h^2+k^2)\mathbf{i} + (ak\ell + bh\ell + chk + hk\ell)\,\mathbf{j}\big\|}{\sqrt{h^2+k^2+\ell^2}} \leqslant \frac{(h^2+k^2) + |ak\ell| + |bh\ell| + |chk| + |hk\ell|}{\sqrt{h^2+k^2+\ell^2}} \\ &\quad \leqslant \sqrt{h^2+k^2+\ell^2} + |a||k| + |b||h| + c|h| + |hk| \end{aligned}$$

thus

$$\lim_{(h,k,\ell)\to(0,0,0)} \frac{\big\|f(a+h,b+k,c+\ell) - f(a,b,c) - L(h,k,\ell)\big\|}{\sqrt{h^2+k^2+\ell^2}} = 0\,.$$

Therefore, f is differentiable at (a,b,c) and $(Df)(a,b,c)$ is given by $(\diamond)$. □

Problem 3. Let $X = \mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)$ equipped with norm $\|\cdot\|$, and $f : \mathrm{GL}(n) \to \mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)$ be defined by $f(L) = L^{-2} \equiv L^{-1}\circ L^{-1}$. Show that f is differentiable on $\mathrm{GL}(n)$ and find $(Df)(L)$ for $L \in \mathrm{GL}(n)$.

Proof. Let $L \in \mathrm{GL}(n)$. By the fact that

$$K^{-1} - L^{-1} = -K^{-1}(K-L)L^{-1} \quad \text{and} \quad K^{-2} - L^{-2} = -K^{-2}(K-L)L^{-1} - K^{-1}(K-L)L^{-2}\,,$$

we have

$$
\begin{aligned}
K^{-2} - L^{-2} &= -\big[L^{-2} - K^{-2}(K-L)L^{-1} - K^{-1}(K-L)L^{-2}\big](K-L)L^{-1} \\
&\quad - \big[L^{-1} - K^{-1}(K-L)L^{-1}\big](K-L)L^{-2} \\
&= -L^{-2}(K-L)L^{-1} - L^{-1}(K-L)L^{-2} + K^{-2}(K-L)L^{-1}(K-L)L^{-1} \\
&\quad + K^{-1}(K-L)L^{-2}(K-L)L^{-1} + K^{-1}(K-L)L^{-1}(K-L)L^{-2}\,;
\end{aligned}
$$

thus

$$
\|K^{-2}-L^{-2}+L^{-2}(K-L)L^{-1}+L^{-1}(K-L)L^{-2}\| \leqslant \Big[\|K^{-2}\|\|L^{-1}\|^2+2\|K^{-1}\|\|L^{-1}\|\|L^{-2}\|\Big]\|K-L\|^2\,. \quad (\star)
$$

This motivates us to define $(Df)(L) \in \mathscr{B}(X,X)$ by

$$
(Df)(L)(H) = -L^{-2}HL^{-1} - L^{-1}HL^{-2} \qquad \forall\, H \in X\,, \tag{$\clubsuit$}
$$

and $(\star)$ implies that

$$
\lim_{K\to L} \frac{\|f(K) - f(L) - (Df)(L)(K-L)\|}{\|K-L\|} = 0\,.
$$

Therefore, f is differentiable on $\mathrm{GL}(n)$, and $(Df)(L)$ is given by $(\clubsuit)$. □

Problem 4. Let $X = \mathscr{C}([-,1,1];\mathbb{R})$ and $\|\cdot\|_X$ be defined by $\|f\|_X = \max\limits_{x\in[-1,1]} \big|f(x)\big|$, and $(Y,\|\cdot\|_Y) = (\mathbb{R},|\cdot|)$. Consider the map $\delta : X \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is differentiable on X. Find $(D\delta)(f)$ (for $f \in \mathscr{C}([-1,1];\mathbb{R})$).

Proof. Let $f \in X$ be given. For $h \in X$, we have

$$
\delta(f+h) - \delta f = \big(f(0) + h(0)\big) - f(0) = h(0) = \delta h\,;
$$

thus we expect that $(D\delta)(f)(h) = \delta h$. We first show that $\delta \in \mathscr{B}(X,\mathbb{R})$.

1. For linearity, for $h_1, h_2 \in X$ and $c \in \mathbb{R}$, we have

$$
\delta(ch_1 + h_2) = \big(ch_1 + h_2\big)(0) = ch_1(0) + h_2(0) = c\delta h_1 + \delta h_2\,.
$$

2. For boundedness, if $\|h\|_X = 1$, then $\max\limits_{x\in[-1,1]} \big|h(x)\big| = 1$ so that

$$
|\delta h| = \big|h(0)\big| \leqslant \max_{x\in[-1,1]} \big|h(x)\big| = 1 < \infty\,.
$$

Having established that $\delta \in \mathscr{B}(X,\mathbb{R})$, we note that

$$
\lim_{h\to 0} \frac{\big|\delta(f+h) - \delta f - \delta h\big|}{\|h\|_X} = \lim_{h\to 0} \frac{0}{\|h\|_X} = 0\,;
$$

thus δ is differentiable at f (for all $f \in X$), and $(D\delta)(f) = \delta$ for all $f \in X$. □

Problem 5. Let $X = \mathscr{C}([a,b];\mathbb{R})$ and $\|\cdot\|_2$ be the norm induced by the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$. Define $I : X \to X$ by

$$I(f)(x) = \int_a^x f(t)^2\,dt \qquad \forall\, x \in [a,b]\,.$$

Show that I is differentiable on X, and find $(DI)(f)$.

Proof. Let $f \in X$ be given. For $h \in X$,

$$I(f+h)(x) - I(f)(x) = \int_a^x \big(f(t)+h(t)\big)^2\,dt - \int_a^x f(t)^2\,dt = \int_a^x \big[2f(t)h(t) + h(t)^2\big]\,dt\,; \qquad (\star\star)$$

thus we expect that

$$(DI)(f)(h)(x) = 2\int_a^x f(t)h(t)\,dt\,. \qquad (\diamond\diamond)$$

Define L by $(Lh)(x) = 2\int_a^x f(t)h(t)\,dt$.
Claim: $L \in \mathscr{B}(X,X)$.

1. For linearity, let $h_1, h_2 \in X$ and $c \in \mathbb{R}$. Then

$$L(ch_1+h_2)(x) = 2\int_a^x f(t)\big(ch_1(t)+h_2(t)\big)\,dt = 2c\int_a^x f(t)h_1(t)\,dt + 2\int_a^x f(t)h_2(t)\,dt$$

which shows that $L(ch_1+h_2) = cL(h_1) + L(h_2)$.

2. Note that by the Cauchy-Schwarz inequality,

$$\Big|\int_a^x f(t)h(t)\,dt\Big| \leqslant \int_a^b |f(t)||h(t)|\,dt \leqslant \|f\|_2\|h\|_2\,;$$

thus for $\|h\|_2 = 1$,

$$\|L(h)\|_2 = \Big[\int_a^b \Big(\int_a^x f(t)h(t)\,dt\Big)^2 dx\Big]^{\frac{1}{2}} \leqslant \Big(\int_a^b \|f\|_2^2\|h\|_2^2\,dx\Big)^{\frac{1}{2}} \leqslant \sqrt{b-a}\|f\|_2\,.$$

Therefore,

$$\|L\| = \sup_{\|h\|_2=1} \|L(h)\|_2 \leqslant \sqrt{b-a}\|f\|_2 < \infty$$

which shows that L is bounded.

Finally, using $(\star\star)$ we obtain that

$$\begin{aligned}\big\|I(f+h) - I(f) - L(h)\big\|_2 &= \Big[\int_a^b \Big(\int_a^x h(t)^2\,dt\Big)^2 dx\Big]^{\frac{1}{2}} \leqslant \Big[\int_a^b \Big(\int_a^b h(t)^2\,dt\Big)^2 dx\Big]^{\frac{1}{2}} \\ &= \Big[\int_a^b \|h\|_2^4\,dx\Big]^{\frac{1}{2}} = \sqrt{b-a}\|h\|_2^2\,;\end{aligned}$$

thus

$$\lim_{h\to 0} \frac{\big\|I(f+h) - I(f) - (DI)(f)(h)\big\|_2}{\|h\|_2} = 0\,.$$

Therefore, I is differentiable at f for all $f \in X$ and $(DI)(f)$ is given by $(\diamond\diamond)$. ◻