Exercise Problem Sets 10

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Problem 1. Check if the following functions on uniformly continuous.

1. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin \log x$.
2. $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x \sin \frac{1}{x}$.
3. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.
4. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x^2)$.
5. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos^3 x$.
6. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin x$.

Problem 2. 1. Find all positive numbers a and b such that the function $f(x) = \dfrac{\sin(x^a)}{1 + x^b}$ is uniformly continuous on $[0, \infty)$.

2. Find all positive numbers a and b such that the function $f(x, y) = |x|^a |y|^b$ is uniformly continuous on $\mathbb{R}^2$.

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and $\lim\limits_{|x| \to \infty} f(x) = b$ exists for some $b \in \mathbb{R}^m$. Show that f is uniformly continuous on $\mathbb{R}^n$.

Proof. Let $\varepsilon > 0$ be given. By the fact that $\lim\limits_{|x| \to \infty} f(x) = b$, there exists $M > 0$ such that

$$\|f(x) - b\|_{\mathbb{R}^m} < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x\|_{\mathbb{R}^n} \geqslant M .$$

By the Heine-Borel Theorem, $B[0, M + 1]$ is compact; thus f is uniformly continuous on $B[0, M + 1]$ and there exists $\delta \in \big(0, \frac{1}{2}\big)$ such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x - y\|_{\mathbb{R}^n} < \delta \text{ and } x, y \in B[0, M + 1] . \tag{$\star$}$$

Therefore, for $x, y \in \mathbb{R}^n$ satisfying $\|x - y\| < \delta$,

1. if $x, y \in B[0, M + 1]$, then $(\star)$ implies that
$$\|f(x) - f(y)\|_{\mathbb{R}^m} < \varepsilon .$$
2. if $x \notin B[0, M + 1]$ or $y \notin B[0, M + 1]$, then $x, y \in B[0, M]^{\complement}$ which implies that
$$\|f(x) - f(y)\|_{\mathbb{R}^m} \leqslant \|f(x)\|_{\mathbb{R}^m} + \|f(y)\|_{\mathbb{R}^m} < \varepsilon . \qquad \square$$

Problem 4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous. Show that there exists $a > 0$ and $b > 0$ such that $\|f(x)\|_{\mathbb{R}^m} \leqslant a\|x\|_{\mathbb{R}^n} + b$.

Proof. Since f is uniformly continuous on $\mathbb{R}^n$, there exists $\delta > 0$ such that

$$\big\|f(x) - f(y)\big\|_{\mathbb{R}^n} < 1 \quad \text{whenever} \quad \|x - y\|_{\mathbb{R}^n} < \delta\,.$$

For a given $x \in \mathbb{R}^n$, let $N \in \mathbb{N}$ such that $\dfrac{\|x\|_{\mathbb{R}^n}}{\delta} < N \leqslant \dfrac{\|x\|_{\mathbb{R}^n}}{\delta} + 1$. For each $k \in \mathbb{N}$, define points x_k by $x_k \equiv \dfrac{kx}{N}$. Then $\{x_k\}_{k=0}^{\infty}$ satisfies that

$$\|x_k - x_{k-1}\|_{\mathbb{R}^m} = \frac{\|x\|_{\mathbb{R}^n}}{N} < \delta \qquad \forall\, k \in \mathbb{N}$$

which further implies that

$$\|f(x_k) - f(x_{k-1})\|_{\mathbb{R}^m} < 1 \qquad \forall\, k \in \mathbb{N}\,.$$

Therefore,

$$\begin{aligned}\|f(x)\|_{\mathbb{R}^m} &\leqslant \|f(x) - f(0)\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} \leqslant \sum_{k=1}^{N} \|f(x_k) - f(x_{k-1})\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} \\ &\leqslant N + \|f(0)\|_{\mathbb{R}^m} \leqslant \frac{1}{\delta}\|x\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} + 1\,;\end{aligned}$$

thus $a = \dfrac{1}{\delta}$ and $b = \|f(0)\|_{\mathbb{R}^m} + 1$ verify the inequality $\|f(x)\|_{\mathbb{R}^m} \leqslant a\|x\|_{\mathbb{R}^n} + b$. □

Problem 5. Let $f(x) = \dfrac{q(x)}{p(x)}$ be a rational function define on $\mathbb{R}$, where p and q are two polynomials. Show that f is uniformly continuous on $\mathbb{R}$ if and only if the degree of q is not more than the degree of p plus 1.

Proof. Note that if f is defined on $\mathbb{R}$, then $p(x) \neq 0$ for all $x \in \mathbb{R}$. By Problem 4, there exist $a, b > 0$ such that

$$\left|\frac{q(x)}{p(x)}\right| \leqslant a|x| + b \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, $|q(x)| \leqslant |p(x)|(a|x| + b)$ for all $x \in \mathbb{R}$, and this inequality above can be true if and only if the degree of q is not more than the degree of p plus 1. □

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function; that is, there exists $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ (and f is continuous). Show that f is uniformly continuous on $\mathbb{R}$.

Proof. Let $p > 0$ be such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$, and $\varepsilon > 0$ be given. Since f is uniformly continuous on $[-p, p]$, there exists $\delta \in (0, p)$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x - y| < \delta \text{ and } x, y \in [-p, p]\,.$$

Therefore, if $|x-y| < \delta$, we must have $x, y \in [kp-p, kp+p]$ for some $k \in \mathbb{Z}$ so that $x-kp, y-kp \in [-p, p]$ which, together with the fact that $|(x - kp) - (y - kp)| = |x - y| < \delta$, implies that

$$\big|f(x) - f(y)\big| = \big|f(x - kp) - f(y - kp)\big| < \varepsilon\,.$$

□

Problem 7. Let $(a,b) \subseteq \mathbb{R}$ be an open interval, and $f : (a,b) \to \mathbb{R}^m$ be a function. Show that the following three statements are equivalent.

1. f is uniformly continuous on (a,b).

2. f is continuous on (a,b), and both limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exist.

3. For all $\varepsilon > 0$, there exists $N > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $\Big|\dfrac{f(x)-f(y)}{x-y}\Big| > N$ and $x, y \in (a,b)$, $x \neq y$.

Proof. First we note that 1 and 2 are equivalent since

1. if f is uniformly continuous on (a,b), then there is a unique continuous extension g of f on $[a,b]$; thus $\lim\limits_{x\to a^+} g(x) = g(a)$ and $\lim\limits_{x\to b^-} g(x) = g(b)$ exists, and 2 holds since $\lim\limits_{x\to a^+} g(x) = \lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} g(x) = \lim\limits_{x\to b^-} f(x)$.

2. if $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exists, we define $g : [a,b] \to \mathbb{R}$ by $g(x) = f(x)$ for $x \in (a,b)$ and $g(a)$, $g(b)$ are respectively the limit of f at a, b. Then g is continuous on $[a,b]$; thus the compactness of $[a,b]$ shows that g is uniformly continuous on $[a,b]$. In particular, g is uniformly continuous on (a,b) which is the same as saying that f is uniformly continuous on (a,b).

Next we prove that 1 and 3 are equivalent.

"$1 \Rightarrow 3$" Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in (a,b)$ such that

$$x_n \neq y_n\,, \quad \big|f(x_n) - f(y_n)\big| \geqslant \varepsilon \quad \text{but} \quad \Big|\frac{f(x_n)-f(y_n)}{x_n-y_n}\Big| > n \qquad \forall\, n \in \mathbb{N}\,.$$

By the Bolzano-Weierstrass Theorem/Property, there exist convergent subsequence $\{x_{n_j}\}_{j=1}^\infty$ and $\{y_{n_j}\}_{j=1}^\infty$ with limit x and y. Since $x_n, y_n \in (a,b)$ for all $n \in \mathbb{N}$, we must have $x, y \in [a,b]$. If $x = y$, then $|x_n - y_n| \to 0$ as $n \to \infty$; thus the uniform continuity of f on (a,b) implies that $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$ which contradicts to the fact that $|f(x_n) - f(y_n)| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. Therefore, $x \neq y$ which further shows that the limit

$$\lim_{n\to\infty} \Big|\frac{f(x_n)-f(y_n)}{x_n-y_n}\Big|$$

exists since the limit $\{f(x_n)\}_{n=1}^\infty$ and $\{f(y_n)\}_{n=1}^\infty$ both exist and $\lim\limits_{n\to\infty}(x_n - y_n) = x - y \neq 0$. This is a contradiction to that $\Big|\dfrac{f(x_n)-f(y_n)}{x_n-y_n}\Big| > n$ for all $n \in \mathbb{N}$.

"$3 \Rightarrow 1$" Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n, y_n \in (a,b)$ satisfying $|x_n - y_n| < \dfrac{1}{n}$ but $|f(x_n) - f(y_n)| \geqslant \varepsilon$. For this $\varepsilon > 0$, by assumption there exists $N > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad \Big|\frac{f(x)-f(y)}{x-y}\Big| > N \text{ and } x, y \in (a,b), x \neq y\,.$$

Since $|f(x_n) - f(y_n)| \geqslant \varepsilon$, we must have $x_n \neq y_n$; thus the fact that $x_n, y_n \in (a, b)$ implies that

$$\left|\frac{f(x_n) - f(y_n)}{x_n - y_n}\right| \leqslant N \qquad \forall\, n \in \mathbb{N}\,.$$

This contradicts to the fact that $|x_n - y_n| < \dfrac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon$. ▫

Problem 8. Suppose that $f : [a, b] \to \mathbb{R}$ is ***Hölder continuous with exponent*** α; that is, there exist $M > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x) - f(y)| \leqslant M|x - y|^\alpha \qquad \forall\, x, y \in [a, b]\,.$$

Show that f is uniformly continuous on $[a, b]$. Show that $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$.

Proof. Let $\varepsilon > 0$ be given. Define $\delta = M^{-\frac{1}{\alpha}}\varepsilon^{\frac{1}{\alpha}}$. Then $\delta > 0$. Moreover, if $|x - y| < \delta$ and $x, y \in [a, b]$,

$$|f(x) - f(y)| \leqslant M|x - y|^\alpha < M\delta^\alpha = \varepsilon\,.$$

Therefore, f is uniformly continuous on $[a, b]$.

Next we show that $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$. Note that if $x, y \geqslant 0$ and $x \neq y$,

$$\frac{|\sqrt{x} - \sqrt{y}|}{|x - y|^{\frac{1}{2}}} = \frac{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|}{|x - y|^{\frac{1}{2}}|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|^{\frac{1}{2}}}{|\sqrt{x} + \sqrt{y}|} \leqslant \frac{\sqrt{x} + \sqrt{y}}{|\sqrt{x} + \sqrt{y}|} \leqslant 1\,;$$

thus

$$|\sqrt{x} - \sqrt{y}| \leqslant |x - y|^{\frac{1}{2}} \qquad \forall\, x, y \geqslant 0 \text{ and } x \neq y\,.$$

which implies that $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$ on $[0, \infty)$. ▫

Problem 9. A function $f : A \times B \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^p$, is said to be separately continuous if for each $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for $y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. f is said to be continuous on A uniformly with respect to B if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big\|f(x, y) - f(x_0, y)\big\|_2 < \varepsilon \text{ whenever } \|x - x_0\|_2 < \delta \text{ and } x \in A\,, y \in B\,.$$

Show that if f is separately continuous and is continuous on A uniformly with respect to B, then f is continuous on $A \times B$.

Proof. Let $\varepsilon > 0$, and $(a, b) \in A \times B$ be given. By assumption there exists $\delta_1 > 0$ such that

$$\big\|f(x, y) - f(a, y)\big\|_2 < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x - a\|_2 < \delta_1 \text{ and } x \in A\,, y \in B\,.$$

Since f is separately continuous, there exists $\delta_2 > 0$ such that

$$\big\|f(a, y) - f(a, b)\big\|_2 < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|y - b\|_2 < \delta_2 \text{ and } y \in B\,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then if $\|(x, y) - (a, b)\|_2 < \delta$, we must have $\|x - a\|_2 < \delta_1$ and $\|y - b\|_2 < \delta_2$ so that

$$\begin{aligned} \|f(x, y) - f(a, b)\|_2 &= \|f(x, y) - f(a, y) + f(a, y) - f(a, b)\|_2 \\ &\leqslant \|f(x, y) - f(a, y)\|_2 + \|f(a, y) - f(a, b)\|_2 < \varepsilon \end{aligned}$$

which shows that f is continuous at (a, b). ▫

Problem 10. Let (M, d) be a metric space, $A \subseteq M$, and $f, g : A \to \mathbb{R}$ be uniformly continuous on A. Show that if f and g are bounded, then fg is uniformly continuous on A. Does the conclusion still hold if f or g is not bounded?

Proof. Let $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ be sequences in A satisfying that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Suppose that $|f(x)| \leqslant M$ and $|g(x)| \leqslant M$ for all $x \in A$. Then

$$\begin{aligned}\big|f(x_n)g(x_n) - f(y_n)g(y_n)\big| &= \big|f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)\big| \\ &\leqslant |f(x_n)|\big|g(x_n) - g(y_n)\big| + |g(y_n)|\big|f(x_n) - f(y_n)\big| \\ &\leqslant M\big(\big|f(x_n) - f(y_n)\big| + \big|g(x_n) - g(y_n)\big|\big)\,;\end{aligned}$$

thus the uniform continuity of f and g, together with the Sandich Lemma, implies that

$$\lim_{n\to\infty} \big|f(x_n)g(x_n) - f(y_n)g(y_n)\big| = 0\,.$$

Therefore, fg is uniformly continuous on A.

When the boundedness is removed from the condition, the product of f and g might not be uniformly continuous. For example, $f(x) = g(x) = x$ are continuous on $\mathbb{R}$, but $(fg)(x) = x^2$ is no uniformly continuous on $\mathbb{R}$ (from an example in class). □

Problem 11. Let $\mathscr{P}([0, 1))$ be the collection of all polynomials defined on $[0, 1]$, and $\|\cdot\|_\infty$ be the max-norm defined by $\|p\|_\infty = \max_{x\in[0,1]} |p(x)|$.

1. Show that the differential operator $\dfrac{d}{dx} : \mathscr{P}([0, 1]) \to \mathscr{P}([0, 1])$ is linear.
2. Show that $\dfrac{d}{dx} : \big(\mathscr{P}([0, 1]), \|\cdot\|_\infty\big) \to \big(\mathscr{P}([0, 1]), \|\cdot\|_\infty\big)$ is unbounded; that is, show that
$$\sup_{\|p\|_\infty=1} \|p'\|_\infty = \infty\,.$$

Proof. 1. Let $p, q \in \mathscr{P}([0, 1])$ and $c \in \mathbb{R}$. Then by the rule of differentiation,

$$\frac{d}{dx}(cp + q)(x) = cp'(x) + q'(x) = c\frac{d}{dx}p(x) + \frac{d}{dx}q(x)\,;$$

thus $\dfrac{d}{dx} : \mathscr{P}([0, 1]) \to \mathscr{P}([0, 1])$ is linear.

2. Consider $p_n(x) = x^n$. Then $\|p_n\|_\infty = \max_{x\in[0,1]} x^n = 1$ for all $n \in \mathbb{N}$; however,

$$\|p_n'\|_\infty = \max_{x\in[0,1]} nx^{n-1} = n \qquad n \in \mathbb{N}\,;$$

thus $\sup_{\|p\|_\infty=1} \|p'\|_\infty = \infty$. □

Problem 12. Recall that $\mathcal{M}_{m\times n}$ is the collection of all $m\times n$ real matrices. For a given $A\in\mathcal{M}_{m\times n}$, define a function $f:\mathcal{M}_{n\times m}\to\mathbb{R}$ by

$$f(M)=\operatorname{tr}(AM)\,,$$

where tr is the trace operator which maps a square matrix to the sum of its diagonal entries. Show that $f\in\mathscr{B}(\mathcal{M}_{n\times m},\mathbb{R})$.

Hint: You may need the conclusion that any two norms on a finite dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$ are equivalent.

Proof. Let $A=[a_{ij}]_{1\leqslant i\leqslant m,1\leqslant j\leqslant n}$ and $M=[m_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}$. Then

$$\operatorname{tr}(AM)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}m_{ji}\,.$$

First we show that $f\in\mathscr{L}(\mathcal{M}_{n\times m},\mathbb{R})$. Let $M=[m_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}$ and $N=[n_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}$ be matrices in $\mathcal{M}_{n\times m}$ and $c\in\mathbb{R}$. Then

$$\begin{aligned}f(cM+N)=\operatorname{tr}(A(cM+N))&=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}(cm_{ji}+n_{ji})=c\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}m_{ji}+\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}n_{ji}\\&=c\operatorname{tr}(AM)+\operatorname{tr}(AN)=cf(M)+f(N)\,.\end{aligned}$$

Let $\|\cdot\|:\mathcal{M}_{n\times m}\to\mathbb{R}$ be defined by

$$\big\|[m_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}\big\|=\sum_{j=1}^{n}\sum_{k=1}^{m}|m_{jk}|\,.$$

Then $\|\cdot\|$ is a norm on $\mathcal{M}_{n\times m}$, and

$$\sup_{\|M\|=1}|f(M)|=\sup_{\sum_{j=1}^{n}\sum_{k=1}^{m}|m_{jk}|=1}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}m_{ji}\Big|\leqslant\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|<\infty\,;$$

thus $f:(\mathcal{M}_{n\times m},\|\cdot\|)\to(\mathbb{R},|\cdot|)$ is bounded. Let $|||\cdot|||$ be another norm on $\mathcal{M}_{n\times m}$. Since $\mathcal{M}_{n\times m}$ is finite dimensional vector spaces over $\mathbb{R}$, there exists c and C such that

$$c\|M\|\leqslant|||M|||\leqslant C\|M\|\qquad\forall\,M\in\mathcal{M}_{n\times m}\,.$$

Therefore, $\big\{M\in\mathcal{M}_{n\times m}\,\big|\,|||M|||\leqslant 1\big\}\subseteq\Big\{M\in\mathcal{M}_{n\times m}\,\Big|\,\|M\|\leqslant\frac{1}{c}\Big\}$

$$\sup_{|||M|||=1}|f(M)|\leqslant\sup_{\|M\|\leqslant 1/c}|f(M)|=\sup_{\|cM\|\leqslant 1}\frac{1}{c}|f(cM)|\leqslant\frac{1}{c}\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|<\infty\,;$$

thus $f:(\mathcal{M}_{n\times m},|||\cdot|||)\to\mathbb{R}$ is bounded. □

Remark 0.1. Problem 12 is a special case of the theorem (about linear maps on a finite dimensional normed space must be bounded) in class.