Exercise Problem Sets 5

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Problem 1. Complete the following.

1. Show that the p-norm on Euclidean space $\mathbb{R}^n$ given by

$$\|x\|_p \equiv \begin{cases} \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \max\big\{|x_1|, \cdots, |x_n|\big\} & \text{if } p = \infty\,, \end{cases} \qquad x = (x_1, \cdots, x_n)$$

is indeed a norm on $\mathbb{R}^n$.

2. Show that for each $1 \leqslant p, q \leqslant \infty$ and $p \neq q$, $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent norms.

Proof. 1. It suffices to show that $\|\cdot\|_p$ satisfies the triangle inequality, and the case of $p = 1$ and $p = \infty$ is left to the readers. First we prove Hölder's inequality: for $1 < p < \infty$,

$$\sum_{i=1}^n a_i b_i \leqslant \Big(\sum_{i=1}^n |a_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^n |b_i|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}.$$

Let $A = \Big(\sum\limits_{i=1}^n |a_i|^p\Big)^{\frac{1}{p}}$ and $B = \Big(\sum\limits_{i=1}^n |b_i|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}$. It suffices to show that

$$\sum_{i=1}^n \frac{a_i}{A}\frac{b_i}{B} \leqslant 1\,.$$

By Young's inequality $ab \leqslant \frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}}$ for all $a, b \geqslant 0$, we find that

$$\frac{a_i}{A}\frac{b_i}{B} \leqslant \frac{1}{p}\Big(\frac{|a_i|}{A}\Big)^p + \frac{p-1}{p}\Big(\frac{|b_i|}{B}\Big)^{\frac{p}{p-1}};$$

thus

$$\sum_{i=1}^n \frac{a_i}{A}\frac{b_i}{B} \leqslant \frac{1}{p}\frac{1}{A^p}\sum_{i=1}^n |a_i|^p + \frac{p-1}{p}\frac{1}{B^{\frac{p}{p-1}}}\sum_{i=1}^n |b_i|^{\frac{p}{p-1}} = \frac{1}{p} + \frac{p-1}{p} = 1\,.$$

Having established Hölder's inequality, we find that

$$\begin{aligned} \|\boldsymbol{x}+\boldsymbol{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \leqslant \sum_{i=1}^n |x_i + y_i|^{p-1}|x_i| + \sum_{i=1}^n |x_i + y_i|^{p-1}|y_i| \\ &\leqslant \Big[\sum_{i=1}^n \big(|x_i + y_i|^{p-1}\big)^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} + \Big[\sum_{i=1}^n \big(|x_i + y_i|^{p-1}\big)^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} \Big(\sum_{i=1}^n |y_i|^p\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{i=1}^n |x_i + y_i|^p\Big)^{\frac{p-1}{p}} \big(\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p\big) = \|\boldsymbol{x}+\boldsymbol{y}\|_p^{p-1}\big(\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p\big)\,. \end{aligned}$$

Therefore, $\|\boldsymbol{x}+\boldsymbol{y}\|_p \leqslant \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

2. It suffices to show that every p-norm is equivalent to the ∞-norm since if so, then for all $1 \leqslant p, q < \infty$ there exist C_1, C_2, C_3, C_4 such that

$$C_1\|\boldsymbol{x}\|_p \leqslant \|\boldsymbol{x}\|_\infty \leqslant C_2\|\boldsymbol{x}\|_p \quad \text{and} \quad C_3\|\boldsymbol{x}\|_q \leqslant \|\boldsymbol{x}\|_\infty \leqslant C_4\|\boldsymbol{x}\|_q \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n\,.$$

Therefore,

$$\frac{C_1}{C_4}\|\boldsymbol{x}\|_p \leqslant \|\boldsymbol{x}\|_q \leqslant \frac{C_2}{C_3}\|\boldsymbol{x}\|_p \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n\,.$$

Now we show that each p-norm is equivalent to the ∞-norm. Note that

$$\|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_p \qquad \forall\, 1 \leqslant p \leqslant \infty\,.$$

On the other hand,

$$\|\boldsymbol{x}\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{i=1}^n \|\boldsymbol{x}\|_\infty^p\Big)^{\frac{1}{p}} \leqslant n^{\frac{1}{p}}\|\boldsymbol{x}\|_\infty\,.$$

Therefore,

$$n^{-\frac{1}{p}}\|\boldsymbol{x}\|_p \leqslant \|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_p \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n \text{ and } 1 \leqslant p \leqslant \infty\,.$$

□

Problem 2. Complete the following.

1. For $f \in \mathscr{C}([a,b];\mathbb{R})$, define

$$\|f\|_p = \begin{cases} \Big(\displaystyle\int_a^b |f(x)|^p\,dx\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \displaystyle\max_{x\in[a,b]} \big|f(x)\big| & \text{if } p = \infty\,. \end{cases}$$

   Show that $\|\cdot\|_p$ is a norm on $\mathscr{C}([a,b];\mathbb{R})$.

2. Are $\|\cdot\|_p$ and $\|\cdot\|_q$ equivalent norms on $\mathscr{C}([a,b];\mathbb{R})$ for any $1 \leqslant p, q \leqslant \infty$?

Proof. 1. For a continuous function $h : [a,b] \to \mathbb{R}$,

$$\int_a^b h(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n h\big(a + i\frac{b-a}{n}\big)\frac{b-a}{n}\,.$$

Therefore, with c_i and d_i denoting $f\big(a+i\frac{b-a}{n}\big)$ and $g\big(a+i\frac{b-a}{n}\big)$, respectively, we have

$$\|f+g\|_p = \lim_{n\to\infty}\Big(\sum_{i=1}^n \Big|(f+g)\big(a+i\frac{b-a}{n}\big)\Big|^p\frac{b-a}{n}\Big)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}}\lim_{n\to\infty}\Big[n^{-\frac{1}{p}}\Big(\sum_{i=1}^n |c_i+d_i|^p\Big)^{\frac{1}{p}}\Big]\,,$$

and similarly,

$$\|f\|_p = (b-a)^{\frac{1}{p}}\lim_{n\to\infty}\Big[n^{-\frac{1}{p}}\Big(\sum_{i=1}^n |c_i|^p\Big)^{\frac{1}{p}}\Big]\,, \qquad \|g\|_p = (b-a)^{\frac{1}{p}}\lim_{n\to\infty}\Big[n^{-\frac{1}{p}}\Big(\sum_{i=1}^n |d_i|^p\Big)^{\frac{1}{p}}\Big]\,.$$

By Minkowski's inequality in Problem 1,

$$n^{-\frac{1}{p}}\Big(\sum_{i=1}^n |c_i+d_i|^p\Big)^{\frac{1}{p}} \leqslant n^{-\frac{1}{p}}\Big(\sum_{i=1}^n |c_i|^p\Big)^{\frac{1}{p}} + n^{-\frac{1}{p}}\Big(\sum_{i=1}^n |d_i|^p\Big)^{\frac{1}{p}}\,;$$

thus the desired conclusion follows from passing to the limit as $n \to \infty$.

2. The 1-norm and the ∞-norm are not equivalent. For each $n \in \mathbb{N}$, consider the function $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} -n^2x + n & \text{if } 0 \leqslant x \leqslant \dfrac{1}{n}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f_n\|_1 = \dfrac{1}{2}$ but $\|f_n\|_\infty = n$. Therefore,

$$\frac{\|f_n\|_\infty}{\|f_n\|_1} = 2n$$

which does not belong to any given bounded interval $[C_1, C_2]$ when n is large. In fact, any p-norm and q-norm cannot be equivalent since for every $n > 0$ one can also find a function $f : [0,1] \to \mathbb{R}$ such that $\|f\|_p = 1$ and $\|f\|_q > n$ if $p < q$. □

Problem 3. Let $\mathcal{M}_{n\times m}$ be the collection of all $n \times m$ real matrices. Define a function $\|\cdot\|_{p,q} : \mathcal{M}_{n\times m} \to \mathbb{R}$ by

$$\|A\|_{p,q} = \sup_{\|\boldsymbol{x}\|_p = 1} \|A\boldsymbol{x}\|_q\,.$$

here we recall that $\|\cdot\|_p$ is the p-norm on Euclidean space. If $p = q$, we simply use $\|A\|_p$ to denote $\|A\|_{p,q}$. Complete the following.

1. Show that $\|A\|_{p,q} = \sup\limits_{\boldsymbol{x}\neq\boldsymbol{0}} \dfrac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p}$ for all $p, q \geqslant 1$.

2. Show that $\|A\|_{p,q} = \inf\big\{M \in \mathbb{R} \,\big|\, \|A\boldsymbol{x}\|_q \leqslant M\|\boldsymbol{x}\|_p \ \forall\, \boldsymbol{x} \in \mathbb{R}^m\big\}$.

3. $\|A\boldsymbol{x}\|_q \leqslant \|A\|_{p,q}\|\boldsymbol{x}\|_p$ for all $\boldsymbol{x} \in \mathbb{R}^m$.

4. Let $\{A_k\}_{k=1}^\infty \subseteq \mathcal{M}_{n\times m}$, and $p, q \geqslant 1$ be given. Show that $\lim\limits_{k\to\infty} \|A_k\|_{p,q} = 0$ if and only if each entry of A_k converges to 0. In other words, by writing $A_k = \big[a_{ij}^{(k)}\big]_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$, show that $\lim\limits_{k\to\infty} \|A_k\|_{p,q} = 0$ if and only if $\lim\limits_{k\to\infty} a_{ij}^{(k)} = 0$ for all $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$.

Proof. 1. If $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{y} = \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|_p}$ satisfies that $\|\boldsymbol{y}\|_p = 1$; thus if $\boldsymbol{x} \neq \boldsymbol{0}$,

$$\frac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p} = \|A\boldsymbol{y}\|_q \leqslant \sup_{\|\boldsymbol{x}\|_p=1} \|A\boldsymbol{x}\|_q = \|A\|_{p,q}\,.$$

Therefore, $\sup\limits_{\boldsymbol{x}\neq\boldsymbol{0}} \dfrac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p} \leqslant \|A\|_{p,q}$.

On the other hand, if $\|\boldsymbol{x}\|_p = 1$, then $\boldsymbol{x} \neq \boldsymbol{0}$; thus if $\|\boldsymbol{x}\|_p = 1$,

$$\|A\boldsymbol{x}\|_q = \frac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p} \leqslant \sup_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p}\,.$$

Therefore, $\|A\|_{p,q} = \sup\limits_{\|\boldsymbol{x}\|_p=1} \|A\boldsymbol{x}\|_q \leqslant \sup\limits_{\boldsymbol{x}\neq\boldsymbol{0}} \dfrac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p}$.

2. 2 follows from Problem 3 in Exercise 2.

3. By 1, $\dfrac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p} \leqslant \|A\|_{p,q}$ for all $\boldsymbol{x} \neq \mathbf{0}$ or equivalently,

$$\|A\boldsymbol{x}\|_q \leqslant \|A\|_{p,q}\|\boldsymbol{x}\|_p \qquad \forall\, \boldsymbol{x} \neq \mathbf{0}\,.$$

Since the inequality above also holds for $\boldsymbol{x} = \mathbf{0}$, we conclude that

$$\|A\boldsymbol{x}\|_q \leqslant \|A\|_{p,q}\|\boldsymbol{x}\|_p \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^m\,.$$

4. Let $B = [b_{ij}] \in M_{n\times m}$, and $|b_{k\ell}| = \max\limits_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m} |b_{ij}|$; that is, the maximum of the absolute value of entries of B occurs at the (k,ℓ)-entry. Let $\boldsymbol{e}_\ell$ be the unit vector whose ℓ-th component is 1. Since $B\boldsymbol{e}_\ell$ is the ℓ-th column of B, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$,

$$|b_{ij}| \leqslant |b_{k\ell}| \leqslant \|B\boldsymbol{e}_\ell\|_q \leqslant \|B\|_{p,q}\|\boldsymbol{e}_\ell\|_p = \|B\|_{p,q}\,;$$

thus

$$|b_{ij}| \leqslant \|B\|_{p,q} \qquad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\,. \tag{$\star$}$$

On the other hand, there exists $\boldsymbol{x} \in \mathbb{R}^m$ such that $\|\boldsymbol{x}\|_p = 1$ and $\|B\boldsymbol{x}\|_q \geqslant \dfrac{\|B\|_{p,q}}{2}$. Therefore, if $1 \leqslant q < \infty$,

$$\begin{aligned}\frac{\|B\|_{p,q}}{2} &\leqslant \|B\boldsymbol{x}\|_q = \Big(\sum_{i=1}^n\Big|\sum_{j=1}^m b_{ij}x_j\Big|^q\Big)^{\frac1q} \leqslant \Big[\sum_{i=1}^n\Big(\sum_{j=1}^m |b_{ij}|\Big)^q\Big]^{\frac1q} \leqslant m\Big[\sum_{i=1}^n\Big(\frac1m\sum_{j=1}^m |b_{ij}|\Big)^q\Big]^{\frac1q}\\ &\leqslant m\Big(\sum_{i=1}^n\frac1m\sum_{j=1}^m |b_{ij}|^q\Big)^{\frac1q} \leqslant m^{1-\frac1q}\Big(\sum_{i=1}^n\sum_{j=1}^m |b_{ij}|^q\Big)^{\frac1q} \leqslant m\Big(\sum_{i=1}^n\sum_{j=1}^m |b_{ij}|^q\Big)^{\frac1q},\end{aligned}$$

while if $q = \infty$,

$$\frac{\|B\|_{p,q}}{2} \leqslant \|B\boldsymbol{x}\|_\infty = \max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^m b_{ij}x_j\Big| \leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^m |b_{ij}| \leqslant \sum_{i=1}^n\sum_{j=1}^m |b_{ij}|\,.$$

In either cases, we conclude that

$$\|B\|_{p,q} \leqslant f(|b_{11}|, |b_{12}|, \cdots, |b_{nm}|) \tag{$\diamond$}$$

for some function f of nm variables satisfying that $f(\boldsymbol{y}) \to 0$ as $\boldsymbol{y} \to 0$.

($\Rightarrow$) Using ($\star$), we find that for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$,

$$0 \leqslant \big|a_{ij}^{(k)}\big| \leqslant \|A_k\|_{p,q}\,.$$

Since $\lim\limits_{k\to\infty} \|A_k\|_{p,q} = 0$, by the Sandwich Lemma we conclude that

$$\lim_{k\to\infty}\big|a_{ij}^{(k)}\big| = 0 \qquad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\,.$$

($\Leftarrow$) Suppose that $\lim\limits_{k\to\infty}\big|a_{ij}^{(k)}\big| = 0$ for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$. Then ($\diamond$) implies that

$$0 \leqslant \|A_k\|_{p,q} \leqslant f\big(\big|a_{11}^{(k)}\big|, \big|a_{12}^{(k)}\big|, \cdots, \big|a_{nm}^{(k)}\big|\big) \tag{$\diamond$}$$

for some function f of nm variables satisfying that $f(\boldsymbol{y}) \to 0$ as $\boldsymbol{y} \to 0$. Therefore, the

Sandwich Lemma implies that $\lim\limits_{k\to\infty} \|A_k\|_{p,q} = 0$. □

Problem 4. Show that

$$\|A\|_1 = \max\left\{ \sum_{i=1}^n |a_{i1}|, \sum_{i=1}^n |a_{i2}|, \cdots, \sum_{i=1}^n |a_{im}| \right\}.$$

Hint: Use Problem 4 and 5 of Exercise 2.

Proof. By Problem 5 of Exercise 2,

$$\|\boldsymbol{x}\|_1 = \sup_{\|\boldsymbol{y}\|_\infty = 1} \boldsymbol{x}\cdot\boldsymbol{y} \qquad \text{and} \qquad \|\boldsymbol{y}\|_\infty = \sup_{\|\boldsymbol{x}\|_1 = 1} \boldsymbol{x}\cdot\boldsymbol{y},$$

where $\boldsymbol{x}\cdot\boldsymbol{y}$ denotes the standard inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ in the Euclidean space. Therefore,

$$\|A\|_1 = \sup_{\|\boldsymbol{x}\|_1=1} \|A\boldsymbol{x}\|_1 = \sup_{\|\boldsymbol{x}\|=1} \sup_{\|\boldsymbol{y}\|_\infty=1} (A\boldsymbol{x})\cdot\boldsymbol{y} = \sup_{\|\boldsymbol{x}\|_1=1} \sup_{\|\boldsymbol{y}\|_\infty=1} \boldsymbol{x}\cdot(A^{\mathrm{T}}\boldsymbol{y}),$$

and Problem 4 of Exercise further implies that

$$\|A\|_1 = \sup_{\|\boldsymbol{y}\|_\infty=1} \sup_{\|\boldsymbol{x}\|_1=1} (A^{\mathrm{T}}\boldsymbol{y})\cdot\boldsymbol{x} = \sup_{\|\boldsymbol{y}\|_\infty=1} \|A^{\mathrm{T}}\boldsymbol{y}\|_\infty = \|A^{\mathrm{T}}\|_\infty.$$

By the fact that the ∞-norm of an $n\times m$ real matrix is the maximum of the sum of the absolute value of entries of row vectors, we find that

$$\|A\|_1 = \|A^{\mathrm{T}}\|_\infty = \max\left\{ \sum_{i=1}^n |a_{i1}|, \sum_{i=1}^n |a_{i2}|, \cdots, \sum_{i=1}^n |a_{im}| \right\}.$$ □

Alternative proof. Let $\boldsymbol{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ and $\|\boldsymbol{x}\|_1 = 1$. Then for $A = [a_{ij}] \in \mathcal{M}_{n\times m}$, we have

$$\begin{aligned}
\|A\boldsymbol{x}\|_1 &= \sum_{i=1}^n \Big| \sum_{j=1}^m a_{ij}x_j \Big| \leqslant \sum_{i=1}^n \sum_{j=1}^m |a_{ij}||x_j| = \sum_{j=1}^m \sum_{i=1}^n |a_{ij}||x_j| = \sum_{j=1}^m |x_j| \Big(\sum_{i=1}^n |a_{ij}| \Big) \\
&\leqslant \sum_{j=1}^m |x_j| \Big(\max_{1\leqslant j\leqslant m} \sum_{i=1}^n |a_{ij}| \Big) = \Big(\max_{1\leqslant j\leqslant m} \sum_{i=1}^n |a_{ij}| \Big) \sum_{j=1}^m |x_j| = \Big(\max_{1\leqslant j\leqslant m} \sum_{i=1}^n |a_{ij}| \Big) \|\boldsymbol{x}\|_1 \\
&= \max_{1\leqslant j\leqslant m} \sum_{i=1}^n |a_{ij}|.
\end{aligned}$$

Therefore, $\|A\|_1 = \sup\limits_{\|\boldsymbol{x}\|_1=1} \|A\boldsymbol{x}\|_1 \leqslant \max\limits_{1\leqslant j\leqslant m} \sum\limits_{i=1}^n |a_{ij}|$.

On the other hand, suppose that $\max\limits_{1\leqslant j\leqslant m} \sum\limits_{i=1}^n |a_{ij}| = \sum\limits_{i=1}^n |a_{ik}|$; that is, the maximum of the sum of absolute value of column entries of A occurs at the k-th column. Let $\boldsymbol{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ be defined by

$$x_j = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Then

$$\|A\boldsymbol{x}\|_1 = \sum_{i=1}^n \Big| \sum_{j=1}^m a_{ij}x_j \Big| = \sum_{i=1}^n |a_{ik}| = \max_{1\leqslant j\leqslant m} \sum_{i=1}^n |a_{ij}|;$$

thus $\|A\|_1 = \sup\limits_{\|\boldsymbol{x}\|_1=1} \|A\boldsymbol{x}\|_1 \geqslant \max\limits_{1\leqslant j\leqslant m} \sum\limits_{i=1}^n |a_{ij}|$. □

Problem 5. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be three normed vector spaces such that $X, Y \subseteq Z$ and

$$\|\boldsymbol{x}\|_Z \leqslant C\|\boldsymbol{x}\|_X \quad \forall\, \boldsymbol{x} \in X \qquad \text{and} \qquad \|\boldsymbol{y}\|_Z \leqslant C\|\boldsymbol{y}\|_Y \quad \forall\, \boldsymbol{y} \in Y\,.$$

Define

$$E = \big\{\boldsymbol{a} \in Z \,\big|\, \|\boldsymbol{a}\|_E \equiv \max\{\|\boldsymbol{a}\|_X, \|\boldsymbol{a}\|_Y\} < \infty\big\}$$

and

$$F = \big\{\boldsymbol{a} \in Z \,\big|\, \|\boldsymbol{a}\|_F \equiv \inf_{\substack{\boldsymbol{a}=\boldsymbol{x}+\boldsymbol{y} \\ \boldsymbol{x}\in X, \boldsymbol{y}\in Y}} (\|\boldsymbol{x}\|_X + \|\boldsymbol{y}\|_Y) < \infty\big\}\,.$$

1. Show that $(E, \|\cdot\|_E)$ is a normed vector space, and $E = X \cap Y$.

2. Show that $(F, \|\cdot\|_F)$ is a normed vector space. The space F is usually denoted by $X + Y$.

Proof. We note that $E, F \subseteq Z$, to show that E and F are vector spaces it suffices to show that E and F are vector subspaces of Z.

1. The case of E: Let $\boldsymbol{a}, \boldsymbol{b} \in E$, and $\lambda \in \mathbb{F}$. Then

$$\max\big\{\|\boldsymbol{a}\|_X, \|\boldsymbol{a}\|_Y\big\} < \infty\,;$$

thus

$$\max\big\{\|\lambda\boldsymbol{a}\|_X, \|\lambda\boldsymbol{a}\|_Y\big\} = |\lambda| \max\big\{\|\boldsymbol{a}\|_X, \|\boldsymbol{a}\|_Y\big\} < \infty \tag{$\star$}$$

which shows that

$$\lambda\boldsymbol{a} \in E \qquad \forall\, \lambda \in \mathbb{F} \text{ and } \boldsymbol{a} \in E\,. \tag{$\diamond$}$$

Moreover,

$$\|\boldsymbol{a}+\boldsymbol{b}\|_X \leqslant \|\boldsymbol{a}\|_X + \|\boldsymbol{b}\|_Y \qquad \text{and} \qquad \|\boldsymbol{a}+\boldsymbol{b}\|_Y \leqslant \|\boldsymbol{a}\|_X + \|\boldsymbol{b}\|_Y$$

which implies that

$$\begin{aligned} \max\big\{\|\boldsymbol{a}+b\|_X, \|\boldsymbol{a}+\boldsymbol{b}\|_Y\big\} &\leqslant \max\big\{\|\boldsymbol{a}\|_X + \|b\|_X, \|\boldsymbol{a}\|_Y + \|\boldsymbol{b}\|_Y\big\} \\ &\leqslant \max\{\|\boldsymbol{a}\|_X, \|\boldsymbol{a}\|_Y\big\} + \max\big\{\|\boldsymbol{b}\|_X, \|\boldsymbol{b}\|_Y\big\} < \infty\,. \end{aligned} \tag{$\star\star$}$$

Therefore,

$$\boldsymbol{a}+\boldsymbol{b} \in E \qquad \forall\, \boldsymbol{a}, \boldsymbol{b} \in E\,. \tag{$\diamond\diamond$}$$

Combining ($\diamond$) and ($\diamond\diamond$), we conclude that

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} \in E \qquad \forall\, \lambda, \mu \in \mathbb{F}, \boldsymbol{a}, \boldsymbol{b} \in E\,;$$

thus Lemma 2.9 shows that E is a subspace of Z.

2. The case of F: Let $\boldsymbol{a}, \boldsymbol{b} \in F$ and $\lambda \in \mathbb{F}$. Then there exists $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in Y$ such that

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{x}_1 + \boldsymbol{y}_1 \qquad \text{and} \qquad \|\boldsymbol{x}_1\|_X + \|\boldsymbol{y}_1\|_Y \leqslant \|\boldsymbol{a}\|_F + 1\,, \\ \boldsymbol{b} &= \boldsymbol{x}_2 + \boldsymbol{y}_2 \qquad \text{and} \qquad \|\boldsymbol{x}_2\|_X + \|\boldsymbol{y}_2\|_Y \leqslant \|\boldsymbol{b}\|_F + 1\,. \end{aligned}$$

Therefore,

$$\lambda \boldsymbol{a} = \lambda \boldsymbol{x}_1 + \lambda \boldsymbol{y}_1 \quad \text{and} \quad \|\lambda \boldsymbol{x}_1\|_X + \|\lambda \boldsymbol{y}_1\|_Y = |\lambda|\big(\|\boldsymbol{x}_1\|_X + \|\boldsymbol{y}_1\|_Y\big) < \lambda\big(\|\boldsymbol{a}\| + 1\big) < \infty$$

which implies that

$$\lambda \boldsymbol{a} \in F \qquad \forall\, \lambda \in \mathbb{F} \text{ and } \boldsymbol{a} \in F\,.$$

Moreover, with $\boldsymbol{x}$ and $\boldsymbol{y}$ denoting $\boldsymbol{x}_1 + \boldsymbol{x}_2$ and $\boldsymbol{y}_1 + \boldsymbol{y}_2$, respectively, we find that $\boldsymbol{x} \in X$, $\boldsymbol{y} \in Y$, $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{x} + \boldsymbol{y}$ and

$$\|\boldsymbol{x}\|_X + \|\boldsymbol{y}\|_Y \leqslant \|\boldsymbol{x}_1\|_X + \|\boldsymbol{x}_2\|_X + \|\boldsymbol{y}_1\|_Y + \|\boldsymbol{y}_2\|_Y < \|\boldsymbol{a}\|_F + \|\boldsymbol{b}\|_F + 2 < \infty\,.$$

This implies that

$$\boldsymbol{a} + \boldsymbol{b} \in F \qquad \forall\, \boldsymbol{a}, \boldsymbol{b} \in F\,.$$

Similar to the case of E, by Lemma 2.9 we conclude that F is a subspace of Z.

Next we show that $\|\cdot\|_E$ and $\|\cdot\|_F$ defined in the problem are indeed norms on E and F, respectively. It is clear that $\|\cdot\|_E$ and $\|\cdot\|_F$ satisfy Property (a) in the definition of the norm vector space, so we only prove Property (b)-(d).

1. The case of E:

   (b) By the definition of $\|\cdot\|_E$,

   $$\|\boldsymbol{a}\|_E = 0 \Leftrightarrow \max\big\{\|\boldsymbol{a}|_X, \|\boldsymbol{a}\|_Y\big\} = 0 \Leftrightarrow \|\boldsymbol{a}|_X = \|\boldsymbol{a}\|_Y = 0 \Leftrightarrow \boldsymbol{a} = \boldsymbol{0}\,.$$

   (c) Let $\lambda \in \mathbb{F}$ and $\boldsymbol{a} \in E$ be given. Then $(\star)$ implies that $\|\lambda \boldsymbol{a}\|_E = |\lambda|\|\boldsymbol{a}\|_E$.

   (d) Let $\boldsymbol{a}, \boldsymbol{b} \in E$. Then $(\star\star)$ implies that $\|\boldsymbol{a} + \boldsymbol{b}\|_E \leqslant \|\boldsymbol{a}\|_E + \|\boldsymbol{b}\|_E$.

   Finally, $a \in E$ if and only if $\|a\|_X < \infty$ and $\|\boldsymbol{a}\|_Y < \infty$; thus $\boldsymbol{a} \in E$ if and only if $\boldsymbol{a} \in X$ and $\boldsymbol{a} \in Y$. This shows that $E = X \cap Y$.

2. The case of F:

   (b) Since $\boldsymbol{0} = \boldsymbol{0} + \boldsymbol{0}$,

   $$\|\boldsymbol{0}\|_F = \inf_{\substack{a = x + y \\ x \in X, y \in Y}} \|\boldsymbol{x}\|_X + \|\boldsymbol{y}\|_Y \leqslant \|\boldsymbol{0}\|_X + \|\boldsymbol{0}\|_Y = 0\,.$$

   Suppose that $\|\boldsymbol{a}\|_F = 0$. For each $n \in \mathbb{N}$, there exists $\boldsymbol{x}_n \in X$ and $\boldsymbol{y}_n \in Y$ such that $\boldsymbol{a} = \boldsymbol{x}_n + \boldsymbol{y}_n$ and

   $$\|\boldsymbol{x}_n\|_X + \|\boldsymbol{y}_n\|_Y < \frac{1}{n}\,.$$

   The inequality above implies that $\boldsymbol{x}_n \to \boldsymbol{0}$ and $\boldsymbol{y}_n \to \boldsymbol{0}$ as $n \to \infty$; thus

   $$\boldsymbol{a} = \lim_{n \to \infty} (\boldsymbol{x}_n + \boldsymbol{y}_n) = \boldsymbol{0}\,.$$

(c) Let $\lambda \in \mathbb{F}$, $\boldsymbol{a} \in E$, and $\varepsilon > 0$ be given. W.L.O.G. we can assume that $\lambda \neq 0$. Then there exists $\boldsymbol{x}_1 \in X$ and $\boldsymbol{y}_1 \in Y$ such that

$$\boldsymbol{a} = \boldsymbol{x}_1 + \boldsymbol{y}_1 \qquad \text{and} \qquad \|\boldsymbol{x}_1\|_X + \|\boldsymbol{y}_1\|_Y < \|\boldsymbol{a}\|_F + \frac{1}{|\lambda|}\varepsilon$$

Then $\lambda\boldsymbol{a} = \lambda\boldsymbol{x}_1 + \lambda\boldsymbol{y}_1$, $\lambda\boldsymbol{x}_1 \in X$, $\lambda\boldsymbol{y}_1 \in Y$ and

$$\|\lambda\boldsymbol{a}\|_F \leqslant \|\lambda\boldsymbol{x}_1\|_X + \|\lambda\boldsymbol{y}_1\|_Y = |\lambda|(\|\boldsymbol{x}_1\|_X + \|\boldsymbol{y}_1\|_Y) < |\lambda|\|\boldsymbol{a}\|_F + \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we find that

$$\|\lambda\boldsymbol{a}\|_F \leqslant |\lambda|\|\boldsymbol{a}\|_F \qquad \forall\,\lambda \in \mathbb{F} \text{ and } \boldsymbol{a} \in F\,.$$

On the other hand, by the fact that $\lambda\boldsymbol{a} \in F$, there exists $\boldsymbol{x}_2 \in X$, $\boldsymbol{y}_2 \in Y$ such that

$$\lambda\boldsymbol{a} = \boldsymbol{x}_2 + \boldsymbol{y}_2 \quad \text{and} \quad \|\lambda\boldsymbol{a}\|_F \leqslant \|\boldsymbol{x}_2\|_X + \|\boldsymbol{y}_2\|_Y + |\lambda|\varepsilon\,.$$

Then $\boldsymbol{a} = \lambda^{-1}\boldsymbol{x}_2 + \lambda^{-1}\boldsymbol{y}_2$, $\lambda^{-1}\boldsymbol{x}_2 \in X$, $\lambda^{-1}\boldsymbol{y}_2 \in Y$, and

$$\|\boldsymbol{a}\|_F \leqslant \|\lambda^{-1}\boldsymbol{x}_2\|_X + \|\lambda^{-1}\boldsymbol{y}_2\|_Y = \frac{1}{|\lambda|}\big(\|\boldsymbol{x}_2\|_X + \|\boldsymbol{y}_2\|_Y\big) < \frac{1}{|\lambda|}\|\lambda\boldsymbol{a}\|_F + \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that $\|\boldsymbol{a}\|_E \leqslant \dfrac{1}{|\lambda|}\|\lambda\boldsymbol{a}\|_F$. Therefore,

$$\|\lambda\boldsymbol{a}\|_F = |\lambda|\|\boldsymbol{a}\|_F \qquad \forall\,\lambda \in \mathbb{F}\,, \boldsymbol{a} \in F\,.$$

(d) Let $\boldsymbol{a}, \boldsymbol{b} \in F$, and $\varepsilon > 0$ be given. There exists $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in Y$ such that

$$\boldsymbol{a} = \boldsymbol{x}_1 + \boldsymbol{y}_1\,, \|\boldsymbol{x}_1\|_X + \|\boldsymbol{y}_1\|_Y < \|\boldsymbol{a}\|_F + \frac{\varepsilon}{2}\,,$$
$$\boldsymbol{b} = \boldsymbol{x}_2 + \boldsymbol{y}_2\,, \|\boldsymbol{x}_2\|_X + \|\boldsymbol{y}_2\|_Y < \|\boldsymbol{b}\|_F + \frac{\varepsilon}{2}\,.$$

Let $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2$ and $\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2$. Then $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$, $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{x} + \boldsymbol{y}$, and

$$\|\boldsymbol{a} + \boldsymbol{b}\|_F \leqslant \|\boldsymbol{x}\|_X + \|\boldsymbol{y}\|_Y \leqslant \|\boldsymbol{x}_1\|_X + \|\boldsymbol{x}_2\|_X + \|\boldsymbol{y}_1\|_Y + \|\boldsymbol{y}_2\|_Y < \|\boldsymbol{a}\|_F + \|\boldsymbol{b}\|_F + \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we find that

$$\|\boldsymbol{a} + \boldsymbol{b}\|_F \leqslant \|\boldsymbol{a}\|_F + \|\boldsymbol{b}\|_F \qquad \forall\,\boldsymbol{a}, \boldsymbol{b} \in F\,.$$

□

Problem 6. Show that if $\langle\cdot,\cdot\rangle$ is an inner product on a vector space $\mathcal{V}$ (over a scalar field $\mathbb{F}$). Then

1. $\langle\lambda\boldsymbol{v} + \mu\boldsymbol{w}, \boldsymbol{u}\rangle = \lambda\langle\boldsymbol{v}, \boldsymbol{u}\rangle + \mu\langle\boldsymbol{w}, \boldsymbol{u}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.

2. $\langle\boldsymbol{u}, \lambda\boldsymbol{v} + \mu\boldsymbol{w}\rangle = \overline{\lambda}\langle\boldsymbol{u}, \boldsymbol{v}\rangle + \overline{\mu}\langle\boldsymbol{u}, \boldsymbol{w}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.

3. $\langle\boldsymbol{v}, \lambda\boldsymbol{w}\rangle = \overline{\lambda}\langle\boldsymbol{v}, \boldsymbol{w}\rangle$ for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.

4. $\langle\boldsymbol{0}, \boldsymbol{w}\rangle = \langle\boldsymbol{w}, \boldsymbol{0}\rangle = 0$ for all $\boldsymbol{w} \in \mathcal{V}$.

Problem 7. Let (M,d) be a metric space. Show that $\rho: M\times M\to\mathbb{R}$ defined by

$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)}$$

is a metric on M.

Proof. Let $x,y,z\in M$.

1. Since $d(x,y)\geqslant 0$, we find that $\rho(x,y)\geqslant 0$.
2. $\rho(x,y)=0 \;\Leftrightarrow\; d(x,y)=0 \;\Leftrightarrow\; x=y$.
3. Since $d(x,y)=d(y,x)$, $\rho(x,y)=\dfrac{d(x,y)}{1+d(x,y)}=\dfrac{d(y,x)}{1+d(y,x)}=\rho(y,x)$.
4. Let $a=d(x,y)$, $b=d(x,z)$ and $c=d(z,y)$. Since $a\leqslant b+c$, we find that

$$\begin{aligned}\rho(x,z)+\rho(z,y)-\rho(x,y)&=\frac{b}{1+b}+\frac{c}{1+c}-\frac{a}{1+a}=\frac{(b+c+2bc)(1+a)-a(1+b+c+bc)}{(1+a)(1+b)(1+c)}\\&=\frac{b+c+2bc+ab+ac+2abc-a-ab-ac-abc}{(1+a)(1+b)(1+c)}\\&=\frac{b+c+2bc+abc-a}{(1+a)(1+b)(1+c)}\geqslant 0\,;\end{aligned}$$

   thus ρ satisfies the triangle inequality. □

Problem 8. Let $d:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be defined by

$$d(\boldsymbol{x},\boldsymbol{y})=\begin{cases}|x_1-y_1| & \text{if } x_2=y_2\,,\\ |x_1-y_1|+|x_2-y_2|+1 & \text{if } x_2\neq y_2\,,\end{cases}\quad \text{where } \boldsymbol{x}=(x_1,x_2) \text{ and } \boldsymbol{y}=(y_1,y_2).$$

Show that d is a metric on $\mathbb{R}^2$.

Proof. Let $\boldsymbol{x}=(x_1,x_2)$, $\boldsymbol{y}=(y_1,y_2)$ and $\boldsymbol{z}=(z_1,z_2)$ in $\mathbb{R}^2$.

1. Clearly $d(\boldsymbol{x},\boldsymbol{y})\geqslant 0$.
2. $d(\boldsymbol{x},\boldsymbol{y})=0\Leftrightarrow(x_2=y_2)\wedge|x_1-y_1|=0\Leftrightarrow(x_2=y_2)\wedge(x_1=y_1)\Leftrightarrow\boldsymbol{x}=\boldsymbol{y}$.
3. (a) The case $x_2=y_2$: In this case $d(\boldsymbol{x},\boldsymbol{y})=|x_1-y_1|$ and $d(\boldsymbol{y},\boldsymbol{x})=|y_1-x_1|$; thus if $x_2=y_2$ then $d(\boldsymbol{x},\boldsymbol{y})=d(\boldsymbol{y},\boldsymbol{x})$.

   (b) The case $x_2\neq y_2$: In this case

$$d(\boldsymbol{x},\boldsymbol{y})=|x_1-y_1|+|x_2-y_2|+1 \quad\text{and}\quad d(\boldsymbol{y},\boldsymbol{x})=|y_1-x_1|+|y_2-x_2|+1\,;$$

   thus if $x_2\neq y_2$ then $d(\boldsymbol{x},\boldsymbol{y})=d(\boldsymbol{y},\boldsymbol{x})$.

   In either cases, we have $d(\boldsymbol{x},\boldsymbol{y})=d(\boldsymbol{y},\boldsymbol{x})$.

4. (a) The case $x_2=y_2$: In this case

$$d(\boldsymbol{x},\boldsymbol{y})=|x_1-y_1|\leqslant|x_1-z_1|+|z_1-y_1|\leqslant d(\boldsymbol{x},\boldsymbol{z})+d(\boldsymbol{z},\boldsymbol{y})\,.$$

(b) The case $x_2 \neq y_2$: In this case z_2 is different from at least one of the second component x_2, y_2. W.L.O.G. we assume that $z_2 \neq x_2$. Then

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{y}) &= |x_1 - y_1| + |x_2 - y_2| + 1 \leqslant |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| + 1 \\ &= d(\boldsymbol{x}, \boldsymbol{z}) + |z_1 - y_1| + |z_2 - y_2| \leqslant d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y})\,. \end{aligned}$$

In either cases, d satisfies the triangle inequality. □