Exercise Problem Sets 1

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Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b \in \mathbb{F}$. Show that $a \leqslant b$ if and only if for all $\varepsilon > 0$, $a < b + \varepsilon$.

Proof. The direction "$\Rightarrow$" is trivial, so we only prove the direction "$\Leftarrow$". Suppose the contrary that $a > b$. Let $\varepsilon = a - b$. Then $\varepsilon > 0$; thus

$$a < b + (a - b) = a\,,$$

a contradiction. □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $x, y \in \mathbb{F}$, and $n \in \mathbb{N}$. Show that

1. If $0 \leqslant x < y$, then $x^n < y^n$.
2. If $0 \leqslant x, y$ and $x^n < y^n$, then $x < y$.

Proof. 1. Let $S = \{n \in \mathbb{N} \,|\, x^n < y^n\}$. Then $1 \in S$ by assumption. Suppose that $n \in S$. Then $0 \leqslant x^n < y^n$. By the fact that $0 \leqslant x < y$, we find that

$$x^{n+1} = x^n \cdot x < x^n \cdot y < y^n \cdot y = y^{n+1}\,;$$

thus $n + 1 \in S$. By induction, we conclude that $S = \mathbb{N}$.

2. Suppose the contrary that $x \geqslant y$. Then 1 implies that $x^n \geqslant y^n$, a contradiction. □

Problem 3. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $I \subseteq \mathbb{F}$ be an interval, and $f : I \to \mathbb{F}$ be a function.

1. f is said to have a limit at c or we say that the limit of f at c exists if

   (a) there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ with limit c, and

   (b) $\lim\limits_{n\to\infty} f(x_n)$ exists for all convergence sequences $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ with limit c.

   Show that the limit of f at c exists if and only if there exists $L \in \mathbb{F}$ satisfying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\,.$$

2. f is said to be continuous at a point $c \in I$ if

$$\lim_{n\to\infty} f(x_n) = f(c) \quad \text{for all convergence sequences } \{x_n\}_{n=1}^{\infty} \subseteq I \text{ with limit } c\,.$$

   Show that f is continuous at c if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - f(c)\big| < \varepsilon \qquad \text{whenever} \quad |x - c| < \delta \text{ and } x \in I\,.$$

Proof. 參考基礎數學第七章影片。 □

Problem 4. Let $(\mathbb{F}, +, \cdot, \leqslant)$ an ordered field satisfying the monotone sequence property, and $y \in \mathbb{F}$ satisfying $y > 1$. Complete the following.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt{y}$ in the last example in class).
2. Show that $y^n - 1 > n(y-1)$ for all $n \in \mathbb{N}\backslash\{1\}$; thus $y - 1 > n(y^{1/n} - 1)$.
3. Show that if $t > 1$ and $n > (y-1)/(t-1)$, then $y^{1/n} < t$.
4. Show that $\lim\limits_{n\to\infty} y^{1/n} = 1$ as $n \to \infty$.

Proof. 1. For each $k \in \mathbb{N}$, let N_k be the largest integer satisfying that $\big(\frac{N_k}{n^k}\big)^n \leqslant y$ but $\big(\frac{N_k+1}{n^k}\big)^n > y$. Define $x_k = \dfrac{N_k}{n^k}$. Then

(a) By binomial expansion, for each $k \in \mathbb{N}$ we have

$$x_k^n \leqslant y < 1 + C_1^n y + C_2^n y^2 + \cdots + C_n^n y^n = (1+y)^n\,;$$

thus Problem 2 implies that $x_k < 1 + y$. Therefore, $\{x_k\}_{k=1}^\infty$ is bounded from above.

(b) For each $k \in \mathbb{N}$,

$$N_k^n \leqslant n^{kn} y \Rightarrow (nN_k)^n \leqslant n^{(k+1)n} y \Rightarrow \big(\frac{nN_k}{n^{k+1}}\big)^n \leqslant y\,;$$

thus $N_{k+1} \geqslant nN_k$. Therefore, for each $k \in \mathbb{N}$,

$$x_k = \frac{N_k}{n^k} = \frac{nN_k}{n^{k+1}} \leqslant \frac{N_{k+1}}{n^{k+1}} = x_{k+1}$$

which shows that $\{x_k\}_{k=1}^\infty$ is increasing.

Therefore, **MSP** implies that $\{x_k\}_{k=1}^\infty$ converges. Assume that $x_k \to x$ as $k \to \infty$ for some $x \in \mathbb{F}$. Then the fact that $x_k^n \leqslant y$ for all $k \in \mathbb{N}$ implies that $x^n \leqslant y$. On the other hand,

$$\Big(x_k + \frac{1}{n^k}\Big)^n \geqslant y \qquad \forall\, k \in \mathbb{N}\,;$$

thus **AP** (a consequence of **MSP**) implies that

$$x^n = \Big(\lim_{k\to\infty} x_k + \lim_{k\to\infty} \frac{1}{n^k}\Big)^n = \lim_{k\to\infty} \Big(x_k + \frac{1}{n^k}\Big)^n \geqslant y\,.$$

Therefore, $x^n = y$. Problem 2 then shows that there is only one $x > 0$ satisfying $x^n = y$. This x will be denoted by $y^{\frac{1}{n}}$.

2. For $y > 1$, let $z = y - 1$. Then $z > 0$ so that for $n > 1$, the binomial expansion shows that

$$\begin{aligned} y^n - 1 &= (1+z)^n - 1 = 1 + C_1^n z + C_2^n z^2 + \cdots + C_n^n z^n - 1 = C_1^n z + C_2^n z^2 + \cdots + C_n^n z^n \\ &> nz = n(y-1)\,. \end{aligned}$$

Therefore, replacing y by $y^{\frac{1}{n}}$ in the inequality above, we conclude that

$$y - 1 > n(y^{\frac{1}{n}} - 1) \qquad \forall\, n \in \mathbb{N}\backslash\{1\}\,.$$

3. Suppose that $y^{\frac{1}{n}} \geqslant t > 1$. Then 2 implies that for $n \in \mathbb{N} \backslash \{1\}$,

$$y - 1 > n(y^{\frac{1}{n}} - 1) \geqslant n(t - 1)\,.$$

   Therefore, $n \leqslant \dfrac{y-1}{t-1}$, a contradiction.

4. Let $k \in \mathbb{N}$ and $t = 1 + \dfrac{1}{k}$ in 3. Then for $n > k(y - 1)$,

$$1 \leqslant y^{\frac{1}{n}} < 1 + \frac{1}{k}\,.$$

   Since $n \to \infty$ as $k \to \infty$, by the Sandwich Lemma we conclude that $\lim_{n\to\infty} y^{\frac{1}{n}} = 1$. $\square$