Exercise Problems for Advanced Calculus

MA2045, National Central University, Fall Semester 2014

§0.1 Sets and Functions

Problem 1. Let S and T be given sets, $A \subseteq S$, $B \subseteq T$, and $f : S \to T$. Show that

1. $f(f^{-1}(B)) \subseteq B$, and $f(f^{-1}(B)) = B$ if $B \subseteq f(S)$.
2. $f^{-1}(f(A)) \supseteq A$, and $f^{-1}(f(A)) = A$ if $f : S \to T$ is one-to-one.

Problem 2. If $f : S \to T$ is a function from S into T, show that the following are equivalent; that is, show that each one of the following implies the other two.

a. f is one-to-one.

b. For every y in T, the set $f^{-1}(\{y\})$ contains at most one point.

c. $f(D_1 \cap D_2) = f(D_1) \cap f(D_2)$ for all subsets D_1 and D_2 of S.

§1.1 Ordered Fields and the Number Systems

Problem 3. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b, c, d \in \mathcal{F}$.

1. Show that if $a \leqslant b$ and $c \leqslant d$, then $a + c \leqslant b + d$.
2. Show that if $a \leqslant b$ and $c < d$, then $a + c < b + d$.

Problem 4. Complete the proof of 11, 12 and 13 Proposition 1.16; that is, show that in an ordered field,

1. If $x \leqslant 0$ and $y \leqslant 0$, then $x \cdot y \geqslant 0$.
2. If $x \leqslant 0$ and $y \geqslant 0$, then $x \cdot y \leqslant 0$.
3. $-1 < 0$.
4. $x^2 \equiv x \cdot x \geqslant 0$ for all $x \in \mathcal{F}$.

Problem 5. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. Show that

1. $|x| \geqslant 0$ for all $x \in \mathcal{F}$.
2. $-|x| \leqslant x \leqslant |x|$ for all $x \in \mathcal{F}$.
3. $|x| = 0$ if and only if $x = 0$.
4. $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathcal{F}$.
5. $|x + y| \leqslant |x| + |y|$ for all $x, y \in \mathcal{F}$.

6. $\big||x| - |y|\big| \leqslant |x - y|$ for all $x, y \in \mathcal{F}$.

Problem 6. Let S be a non-empty subset of $\mathbb{N}$ and satisfy that

1. $1,\ 2 \in S$.

2. if m and $m + 1 \in S$, then $m + 2 \in S$.

Show that $S = \mathbb{N}$.

Problem 7. 1. Let S be a non-empty set. Show that S is countable if and only if there exists a surjection $f : \mathbb{N} \to \mathbb{S}$.

2. Let S be a non-empty set, and A be a non-empty subset of S. Show that there exists a surjection $g : S \to A$.

3. Use 1 and 2 to show that any non-empty subset of a countable set is countable.

4. Let S be a non-empty set. Show that S is countable if and only if there exists a injection $f : S \to \mathbb{N}$.

§1.2 Completeness and the Real Number System

Problem 8. Let $\mathcal{F}$ be an ordered field with Archimedean property, and $x, y \in \mathcal{F}$. Show that $x \leqslant y$ if and only if $\forall\, \varepsilon > 0$, $x < y + \varepsilon$.

Problem 9. Let $\mathcal{F}$ be a complete ordered field, $y \in \mathcal{F}$ and $y > 1$.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt{y}$ in class).

2. Show that $y^n - 1 > n(y - 1)$ for all $n \in \mathbb{N}$; thus $y - 1 > n(y^{1/n} - 1)$.

3. Show that if $t > 1$ and $n > (y - 1)/(t - 1)$, then $y^{1/n} < t$.

4. Show that $\lim\limits_{n\to\infty} y^{1/n} = 1$ as $n \to \infty$.

What can you conclude if $y < 1$?

Problem 10. Let x_n be a monotone increasing sequence in a complete ordered field such that $x_{n+1} - x_n \leqslant \dfrac{1}{n}$. Must x_n converge? How about if $x_{n+1} - x_n \leqslant \dfrac{1}{2^n}$?

§1.3 Least Upper Bounds

Problem 11. Let A be a non-empty set of $\mathbb{R}$ which is bounded below. Define the set $-A$ by $-A \equiv \big\{ -x \in \mathbb{R} \,\big|\, x \in A \big\}$. Prove that

$$\inf A = -\sup(-A)\,.$$

Problem 12. Let A, B be non-empty subset of $\mathbb{R}$. Define $A + B = \{x + y \,|\, x \in A, y \in B\}$. Justify if the following statements are true or false by providing a proof for the true statement and giving a counter-example for the false ones.

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(A + B) = \inf A + \inf B$.
3. $\sup(A \cap B) \leqslant \min\{\sup A, \sup B\}$.
4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
5. $\sup(A \cup B) \geqslant \max\{\sup A, \sup B\}$.
6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Problem 13. Let $S \subseteq \mathbb{R}$ be bounded below and non-empty. Show that

$$\inf S = \sup \left\{x \in \mathbb{R} \,\middle|\, x \text{ is a lower bound for } S\right\}.$$

Problem 14. Fix $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that
   (a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.
   (b) if r, s in $\mathbb{Q}$, then $b^{r \cdot s} = (b^r)^s$.
2. For $x \in \mathbb{R}$, let $B(x) = \left\{b^t \in \mathbb{R} \,\middle|\, t \in \mathbb{Q}, t \leqslant x\right\}$. Show that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. Therefore, it makes sense to define $b^x = \sup B(x)$ for $x \in \mathbb{R}$. Show that the law of exponents (for real exponents) are also valid.
3. Let $y > 0$ be given. Using 4 of Problem 9 to show that if $u, v \in \mathbb{R}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n.
4. Let $y > 0$ be given, and A be the set of all w such that $b^w < y$. Show that $x = \sup A$ satisfies $b^x = y$.
5. Prove that if x_1, x_2 are two real numbers satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.

The number x satisfying $b^x = y$ is called the logarithm of y to the base b, and is denoted by $\log_b y$.

Problem 15. Prove or disprove the following statement: let $A \subseteq \mathbb{R}$ satisfy

$$\sup\left\{\sum_{b \in B} |b| \,\middle|\, B \text{ is a non-empty finite subset of } A\right\} < \infty.$$

Then $\left\{x \in A \,\middle|\, x \neq 0\right\}$ is countable.

§1.4 Cauchy Sequences

Problem 16. Let $\mathcal{F}$ be an ordered field, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall\, \varepsilon > 0, \exists\, y \in \mathcal{F} \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (y-\varepsilon, y+\varepsilon)\big\} < \infty\,.$$

Problem 17. Let $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$, and define $S_k = \sum\limits_{n=1}^{k} a_n$ (so $\{S_k\}_{k=1}^{\infty}$ is also a sequence). Suppose that $|x_n - x_{n+1}| < a_n$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^{\infty}$ converges if $\{S_k\}_{k=1}^{\infty}$ converges.

Problem 18. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two Cauchy sequence in $\mathbb{R}$. Show that the sequence $\big\{|x_n - y_n|\big\}_{n=1}^{\infty}$ converges.

Problem 19. True or false. Provide a proof if the statement is true, and provide a counter-example if the statement is wrong.

1. If a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ satisfies $x_{n+1} - \epsilon_n \leqslant x_n$ for $n \in \mathbb{N}$, where $\sum\limits_{n=1}^{\infty} \epsilon_n$ is an absolute convergent series; that is, the partial sum $\sum\limits_{n=1}^{k} |\epsilon_n|$ converges as $k \to \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges.

2. Let $\pi : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto (such map π is called a rearrangement), and $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence. Then $\big\{x_{\pi(n)}\big\}_{n=1}^{\infty}$ is also convergent.

§1.5 Cluster Points; lim inf and lim sup

Problem 20. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leqslant \liminf_{n\to\infty}(x_n + y_n) \leqslant \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
&\leqslant \limsup_{n\to\infty}(x_n + y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\,; \\
(\liminf_{n\to\infty} |x_n|)(\liminf_{n\to\infty} |y_n|) &\leqslant \liminf_{n\to\infty} |x_n y_n| \leqslant (\liminf_{n\to\infty} |x_n|)(\limsup_{n\to\infty} |y_n|) \\
&\leqslant \limsup_{n\to\infty} |x_n y_n| \leqslant (\limsup_{n\to\infty} |x_n|)(\limsup_{n\to\infty} |y_n|)\,.
\end{aligned}$$

Give examples showing that the equalities are generally not true.

Problem 21. Prove that

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}\,.$$

Give examples to show that the equalities are not true in general. Is it true that $\lim\limits_{n\to\infty} \sqrt[n]{|x_n|}$ exists implies that $\lim\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$ also exists?

Problem 22. Given the following sets consisting of elements of some sequence of real numbers. Find their sup and inf, and also the limsup and liminf of the sequence.

1. $\{\cos m \,|\, m = 0, 1, 2, \cdots\}$.
2. $\{(1 + \frac{1}{m}) \sin \frac{m\pi}{6} \,|\, m = 1, 2, \cdots\}$.

Hint: For 1, first show that for all irrational α, the set

$$S = \{x \in [0,1] \,|\, x = k\alpha \pmod 1 \text{ for some } k \in \mathbb{N}\}$$

is dense in $[0,1]$; that is, for all $y \in [0,1]$ and $\varepsilon > 0$, there exists $x \in S \cap (y - \varepsilon, y + \varepsilon)$. Then choose $\alpha = \frac{1}{2\pi}$ to conclude that

$$T = \{x \in [0, 2\pi] \,|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\}$$

is dense in $[0, 2\pi]$. To prove that S is dense in $[0,1]$, you might want to consider the following set

$$S_k = \{x \in [0,1] \,|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\}$$

Note that there must be two points in S_k whose distance is less than $\frac{1}{k}$. What happened to (the multiples of) the difference of these two points?

§1.6 Euclidean Space

Problem 23. Show that the p-norm on Euclidean space $\mathbb{R}^n$ given by

$$\|x\|_p \equiv \begin{cases} \Big(\sum\limits_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \max\{|x_1|, \cdots, |x_n|\} & \text{if } p = \infty\,, \end{cases} \qquad x = (x_1, \cdots, x_n)$$

is indeed a norm.

§1.7 Norms, Inner Products, and Metrics

Problem 24. Let $\mathcal{M}$ be the collection of all $n \times m$ matrices with real entries. Define a function $\|\cdot\| : \mathcal{M} \to \mathbb{R}$ by

$$\|A\| = \sup_{\substack{x \in \mathbb{R}^m \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}\,,$$

here we recall that $\|\cdot\|_2$ is the 2-norm on Euclidean space given by

$$\|x\|_2 = \Big(\sum_{i=1}^k x_i^2\Big)^{1/2} \quad \text{if} \quad x = (x_1, \cdots, x_k) \in \mathbb{R}^k\,.$$

Show that

1. $\|A\| = \sup\limits_{\substack{x \in \mathbb{R}^m \\ \|x\|_2 = 1}} \|Ax\|_2 = \inf\{M \in \mathbb{R} \,|\, \|Ax\|_2 \leqslant M\|x\|_2 \ \forall\, x \in \mathbb{R}^m\}$.

2. $\|Ax\|_2 \leqslant \|A\|\|x\|_2$ for all $x \in \mathbb{R}^m$.

3. $\|\cdot\|$ defines a norm on $\mathcal{M}$.

Problem 25. Let $(\mathcal{V}, +, \cdot, \langle\cdot,\cdot\rangle)$ be an inner product space, and define $\|v\| = \langle v, v\rangle^{1/2}$ for all $v \in \mathcal{V}$. Show that

1. $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ (parallelogram law).

2. $\|x+y\|\|x-y\| \leqslant \|x\|^2 + \|y\|^2$.

3. $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ (polarization identity).

Can the p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ be induced from any inner product (on $\mathbb{R}^n$) for $p \neq 2$?

Problem 26. Let (M, d) be a metric space. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)} \,.$$

Show that (M, ρ) is also a metric space.

§2.1 Open Sets

Problem 27. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $\mathcal{U} \subseteq \mathbb{R}$ is open, then

$$\mathcal{U} = \bigcup_{k \in \mathcal{I}} (a_k, b_k) \,,$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.

Problem 28. Let (M, d) be a metric space, and $A \subseteq M$. An open cover of A is a collection of open sets whose union contains A; that is, $\{\mathcal{U}_i\}_{i \in \mathcal{I}}$ is called an open cover of A if

1. $\mathcal{U}_i$ is open for all $i \in \mathcal{I}$.

2. $A \subseteq \bigcup\limits_{i \in \mathcal{I}} \mathcal{U}_i$.

Show that

1. if $\big\{(a_k, b_k)\big\}_{k=1}^\infty$ is an open cover of $[a, b] \subseteq \mathbb{R}$, then there exists $N > 0$ such that $\bigcup\limits_{k=1}^{N} (a_k, b_k) \supseteq [a, b]$.

2. Using Exercise 27 to show that if $\{\mathcal{U}_k\}_{k=1}^\infty$ is an open cover of $[a, b]$, then there exists $N > 0$ such that $\bigcup\limits_{k=1}^{N} \mathcal{U}_k \supseteq [a, b]$.

§2.2 Interior of a set

Problem 29. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{int}(\text{int}(A)) = \text{int}(A)$.
2. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

§2.3 Closed Sets, §2.4 Accumulation Points, Limit Points, and Isolated Points

Problem 30. Let (M, d) be a metric space, and $A \subseteq M$. Show (by definition) that $\bar{A}$ is closed .

Problem 31. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Suppose $\{x_n\}_{n=1}^{\infty} \subseteq A$ is a convergent sequence with values in A. Show that the limit of $\{x_n\}_{n=1}^{\infty}$ belongs to $\bar{A}$.

Problem 32. True or false. Provide a proof if the statement is true, and provide a counter-example if the statement is wrong.

1. An interior point of a subset A of a metric space (M, d) is an accumulation point of that set.
2. Let (M, d) be a metric space, and $A \subseteq M$. Then $(A')' = A'$.

Problem 33. Let (M, d) be a metric space, and $A \subseteq M$. Show that $A' = \bar{A} \backslash (A \backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A} \backslash A' = A \backslash A'$.

§2.5 Closure of Sets

Problem 34. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.
2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

§2.6 Boundary of Sets

Problem 35. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Show that

$$\partial A = \big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big).$$

Problem 36. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial A = \partial(M \backslash A)$.
2. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$.
3. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.

4. If $\text{cl}(A) \cap \text{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.

5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Problem 37. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Determine which of the following statements are true.

1. $\text{int}A = A \backslash \partial A$.

2. $\text{cl}(A) = M \backslash \text{int}(M \backslash A)$.

3. $\text{int}(\text{cl}(A)) = \text{int}(A)$.

4. $\text{cl}(\text{int}(A)) = A$.

5. $\partial(\text{cl}(A)) = \partial A$.

6. If A is open, then $\partial A \subseteq M \backslash A$.

7. If A is open, then $A = \text{cl}(A) \backslash \partial A$. How about if A is not open?

Problem 38. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be perfect if $A = A'$ (that is, A has no isolated points). The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the intervals

$$\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the Cantor set.

1. Show that C is a perfect set.

2. Show that C is uncountable.

3. Find $\text{int}(C)$.

Problem 39. Complete the following.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.

2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \text{cl}(A \cap B)$.

§2.7 Sequences, §2.8 Completeness

Problem 40. Let (M, d) be a metric space, and $N \subseteq M$. Show that if (N, d) is complete, then N is closed.

Remark: In class we have shown that if (M, d) is a complete metric space, and N is a closed subset of M, then (N, d) is complete. This problem gives a reverse statement.

Problem 41. (本題期中考不考，有興趣的同學自己練習) Let (M, d) be a metric space. Call two Cauchy sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ in M equivalent, denoted by $\{p_n\}_{n=1}^{\infty} \sim \{q_n\}_{n=1}^{\infty}$, if

$$\lim_{n\to\infty} d(p_n, q_n) = 0\,.$$

1. Prove that $\sim$ is an equivalence relation; that is, show that
   (a) $\{p_n\}_{n=1}^{\infty} \sim \{p_n\}_{n=1}^{\infty}$.
   (b) If $\{p_n\}_{n=1}^{\infty} \sim \{q_n\}_{n=1}^{\infty}$, then $\{q_n\}_{n=1}^{\infty} \sim \{p_n\}_{n=1}^{\infty}$.
   (c) If $\{p_n\}_{n=1}^{\infty} \sim \{q_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty} \sim \{r_n\}_{n=1}^{\infty}$, then $\{p_n\}_{n=1}^{\infty} \sim \{r_n\}_{n=1}^{\infty}$.
2. Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be two Cauchy sequence, show that the sequence $\big\{d(p_n, q_n)\big\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$; thus is convergent.
3. Let M^* be the set of all equivalence classes. If $P, Q \in M^*$, we define

   $$d^*(P, Q) = \lim_{n\to\infty} d(p_n, q_n)\,,$$

   where $\{p_n\}_{n=1}^{\infty} \in P$ and $\{q_n\}_{n=1}^{\infty} \in Q$. Show that the definition above is well-defined; that is, show that f $\{p'_n\}_{n=1}^{\infty} \in P$ and $\{q'_n\}_{n=1}^{\infty} \in Q$ are another two Cauchy sequences, then $\lim\limits_{n\to\infty} d(p_n, q_n) = \lim\limits_{n\to\infty} d(p'_n, q'_n)$.
4. Define $\varphi : M \to M^*$ as follows: for each $x \in M$, $\{x_n\}_{n=1}^{\infty}$, where $x_n \equiv x$ for all $n \in \mathbb{N}$, is a Cauchy sequence in M. Then $\{x_n\}_{n=1}^{\infty} \in \varphi(x)$ for one particular $\varphi(x) \in M^*$. In other words, $\varphi(x)$ is the equivalence class where $\{x_n\}_{n=1}^{\infty}$ belongs to. Show that

   $$d^*\big(\varphi(x), \varphi(y)\big) = d(x, y) \qquad \forall\, x, y \in M.$$

5. Show that $\varphi(M)$ is dense in M^*.
6. Show that (M^*, d^*) is a complete metric space. The metric space (M^*, d^*) is called the completion of (M, d).

§2.9 Series of Real Numbers and Vectors

Problem 42. Prove the root test and the alternative series test in Theorem 2.88 of the lecture note.

§3.1 Compactness

Problem 43. Let (M, d) be a metric space.

1. Show that the union of a finite number of compact subsets of M is compact.

2. Show that the intersection of an arbitrary collection of compact subsets of M is compact.

Problem 44. A metric space (M, d) is said to be separable if there is a countable subset A which is dense in M. Show that every compact set is separable.

Problem 45. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

1. Show that d is a metric on $\mathbb{R}^2$. In other words, $(\mathbb{R}^2, d)$ is a metric space.

2. Find $D(x, r)$ with $r < 1$, $r = 1$ and $r > 1$.

3. Show that the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is closed and bounded.

4. Examine whether the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is compact or not.

Problem 46. Let (M, d) be a complete metric space, and $A \subseteq M$ be totally bounded. Show that $\text{cl}(A)$ is compact.

Problem 47. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space, and $x_k \to x$ as $k \to \infty$. Show that the set $A \equiv \{x_1, x_2, \cdots, \} \cup \{x\}$ is compact by

1. showing that A is sequentially compact; and

2. showing that every open cover of A has a finite subcover; and

3. showing that A is totally bounded and complete.

Problem 48. Let (M, d) be a metric space, $K \subseteq M$ be compact, and $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of K. Show that there exists $r > 0$ such that if $x \in K$, then $D(x, r) \subseteq \mathcal{U}_\alpha$ for some $\alpha \in I$.

Remark. The supremum of all such $r > 0$ is called the Lebesgue number for the cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$.

Problem 49. Prove Theorem 3.24 in the lecture note; that is, show that if (M, d) is a metric space, and $K \subseteq M$, then K is compact if and only if every collection of closed sets with the finite intersection property for K has non-empty intersection with K.

Problem 50. Let X be the collection of all sequences $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that $\sup\limits_{k \geqslant 1} |x_k| < \infty$. In other words,

$$X = \Big\{ \{x_k\}_{k=1}^{\infty} \,\Big|\, x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \sup_{k \geqslant 1} |x_k| < \infty \Big\}.$$

Define $\|\cdot\| : X \to \mathbb{R}$ by

$$\big\| \{x_k\}_{k=1}^{\infty} \big\| = \sup_{k \geqslant 1} |x_k|.$$

1. Show that $\|\cdot\|$ is a norm on X. The normed space $(X, \|\cdot\|)$ usually is denoted by ℓ^{∞}.

2. Show that $(X, \|\cdot\|)$ is complete.

3. Let A, B, C, D be a subsets of X given by

$$\begin{aligned} A &= \Big\{ \{x_k\}_{k=1}^{\infty} \,\Big|\, |x_k| \leqslant \frac{1}{k} \text{ for all } k \in \mathbb{N} \Big\}, \\ B &= \Big\{ \{x_k\}_{k=1}^{\infty} \,\Big|\, x_k \to 0 \text{ as } k \to \infty \Big\}, \\ C &= \Big\{ \{x_k\}_{k=1}^{\infty} \,\Big|\, \text{the sequence } \{x_k\}_{k=1}^{\infty} \text{ converges} \Big\}, \\ D &= \Big\{ \{x_k\}_{k=1}^{\infty} \,\Big|\, \sup_{k \geqslant 1} |x_k| = 1 \Big\}. \end{aligned}$$

   Determine whether A, B, C, D are compact or not.

Problem 51. Let Y be the collection of all sequences $\{y_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} |y_k|^2 < \infty$. In other words,

$$Y = \Big\{ \{y_k\}_{k=1}^{\infty} \,\Big|\, y_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \sum_{k=1}^{\infty} |y_k|^2 < \infty \Big\}.$$

Define $\|\cdot\| : Y \to \mathbb{R}$ by

$$\big\| \{y_k\}_{k=1}^{\infty} \big\| = \Big(\sum_{k=1}^{\infty} |y_k|^2 \Big)^{\frac{1}{2}}.$$

1. Show that $\|\cdot\|$ is a norm on Y. The normed space $(Y, \|\cdot\|)$ usually is denoted by ℓ^2.

2. Show that $\|\cdot\|$ is induced by an inner product.

3. Show that $(Y, \|\cdot\|)$ is complete.

4. Let $E = \big\{ y \in Y \,\big|\, \|y\| \leqslant 1 \big\}$. Is E compact or not?

Problem 52. Let A, B be two non-empty subsets in $\mathbb{R}^n$. Define

$$d(A, B) = \inf \big\{ \|x - y\|_2 \,\big|\, x \in A, y \in B \big\}$$

to be the distance between A and B. When $A = \{x\}$ is a point, we write $d(A, B)$ as $d(x, B)$.

(1) Prove that $d(A, B) = \inf \big\{ d(x, B) \,\big|\, x \in A \big\}$.

(2) Show that $\big| d(x_1, B) - d(x_2, B) \big| \leqslant \|x_1 - x_2\|_2$ for all $x_1, x_2 \in \mathbb{R}^n$.

(3) Define $B_\varepsilon = \{x \in \mathbb{R}^n \,|\, d(x, B) < \varepsilon\}$ be the collection of all points whose distance from B is less than ε. Show that B_ε is open and $\bigcap_{\varepsilon>0} B_\varepsilon = \mathrm{cl}(B)$.

(4) If A is compact, show that there exists $x \in A$ such that $d(A, B) = d(x, B)$.

(5) If A is closed and B is compact, show that there exists $x \in A$ and $y \in B$ such that $d(A, B) = d(x, y)$.

(6) If A and B are both closed, does the conclusion of (5) hold?

§3.2 The Heine-Borel Theorem

Problem 53. Let $M = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ with the standard metric $\|\cdot\|_2$. Show that $A \subseteq M$ is compact if and only if A is closed.

§3.3 Nested Set Property

Problem 54. 1. Let $\{x_k\}_{k=1}^\infty \subseteq \mathbb{R}$ be a sequence in $(\mathbb{R}, |\cdot|)$ that converges to x and let $A_k = \{x_k, x_{k+1}, \cdots\}$. Show that $\{x\} = \bigcap_{k=1}^\infty \overline{A_k}$. Is this true in any metric space?

2. Suppose that $\{K_j\}_{j=1}^\infty$ is a sequence of comapct non-empty sets satisfying the nested set property; that is, $K_j \supseteq K_{j+1}$, and diameter$(K_j) \to 0$ as $j \to \infty$, where

$$\text{diameter}(K_j) = \sup\{d(x, y) \,|\, x, y \in K_j\}.$$

Show that there is exactly one point in $\bigcap_{j=1}^\infty K_j$.

§3.4 Connectedness

Problem 55. Let (M, d) be a metric space, and $A \subseteq M$. Show that A is disconnected (not connected) if and only if there exist non-empty closed set F_1 and F_2 such that

1. $A \cap F_1 \cap F_2 = \varnothing$; 2. $A \cap F_1 \neq \varnothing$; 3. $A \cap F_2 \neq \varnothing$; 4. $A \subseteq F_1 \cup F_2$.

Problem 56. Prove that if A is connected in a metric space (M, d) and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Problem 57. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Suppose that A is connected and contain more than one point. Show that $A \subseteq A'$.

Problem 58. Show that the Cantor set C defined in Problem 38 is totally disconnected; that is, if $x, y \in C$, and $x \neq y$, then $x \in \mathcal{U}$ and $y \in \mathcal{V}$ for some open sets $\mathcal{U}$, $\mathcal{V}$ separate C.

Problem 59. Let F_k be a nest of connected compact sets (that is, $F_{k+1} \subseteq F_k$ and F_k is connected for all $k \in \mathbb{N}$). Show that $\bigcap_{k=1}^\infty F_k$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that F_k is a nest of closed connected sets.

§4.1 Continuity

Started from this section, for all $n \in \mathbb{N}$ $\mathbb{R}^n$ always denotes the normed space $(\mathbb{R}^n, \|\cdot\|_2)$.

Problem 60. Complete the following.

1. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x, y)$$

   exist but are not equal.

2. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits above exist and are equal but f is not continuous.

3. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous.

Problem 61. Complete the following.

1. Show that the projection map $f : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x \end{array}$ is continuous.

2. Show that if $\mathcal{U} \subseteq \mathbb{R}$ is open, then $A = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x \in \mathcal{U}\right\}$ is open.

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $\mathcal{U} \subseteq \mathbb{R}$ such that $f(\mathcal{U})$ is not open.

Problem 62. Show that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is continuous if and only if for every $B \subseteq A$,

$$f(\text{cl}(B) \cap A) \subseteq \text{cl}(f(B)).$$

§4.2 Images of Compact and Connected Sets under Continuous Mappings

Problem 63. Complete the following.

1. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $B \subseteq \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact, is $f^{-1}(K)$ necessarily compact?

3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $C \subseteq \mathbb{R}$ is connected, is $f^{-1}(C)$ necessarily connected?

Problem 64. Consider a compact set $K \subseteq \mathbb{R}^n$ and let $f : K \to \mathbb{R}^m$ be continuous and one-to-one. Show that the inverse function $f^{-1} : f(K) \to K$ is continuous. How about if K is not compact but connected?

§4.6 Uniform Continuity

Problem 65. Check if the following functions on uniformly continuous.

1. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin \log x$.
2. $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x \sin \dfrac{1}{x}$.
3. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.

Problem 66. A function $f : A \times B \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^p$, is said to be separately continuous if for each $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for $y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. f is said to be continuous on A uniformly with respect to B if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \|f(x, y) - f(x_0, y)\|_2 < \varepsilon \text{ whenever } \|x - x_0\|_2 < \delta \text{ and } y \in B\,.$$

Show that if f is separately continuous and is continuous on A uniformly with respect to B, then f is continuous on $A \times B$.

Problem 67. Complete the following.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function; that is, $\exists\, p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ (and f is continuous). Show that f is uniformly continuous on $\mathbb{R}$.

2. Suppose that $a, b \in \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ is continuous. Show that f is uniformly continuous on (a, b) if and only if the two limits

   $$\lim_{x \to a^+} f(x) \quad \text{and} \quad \lim_{x \to b^-} f(x)$$

   exist. How about if (a, b) is not a finite interval?

3. Suppose that $f : [a, b] \to \mathbb{R}$ is ***Hölder continuous with exponent*** α; that is, there exist $M > 0$ and $\alpha \in (0, 1]$ such that

   $$|f(x_1) - f(x_2)| \leqslant M|x_1 - x_2|^\alpha \qquad \forall\, x_1, x_2 \in [a, b]\,.$$

   Show that f is uniformly continuous on $[a, b]$. Show that $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$.

Problem 68. Let (M, d) be a metric space, $A \subseteq M$, and $f, g : A \to \mathbb{R}$ be uniformly continuous on A. Show that if f and g are bounded, then fg is uniformly continuous on A. Does the conclusion still hold if f or g is not bounded?

§4.7 Differentiation of Functions of One Variable

Problem 69. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, and $f \geqslant 0$. Find $\dfrac{d}{dx} f(x)^{g(x)}$.

Problem 70. Suppose α and β are real numbers, $\beta > 0$ and $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^\alpha \sin(x^{-\beta}) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Prove the following statements.

1. f is continuous if and only if $\alpha > 0$.
2. $f'(0)$ exists if and only if $\alpha > 1$.
3. f' is bounded if and only if $\alpha \geqslant 1 + \beta$.
4. f' is continuous if and only if $\alpha > 1 + \beta$.
5. $f''(0)$ exists if and only if $\alpha > 2 + \beta$.
6. f'' is bounded if and only if $\alpha \geqslant 2 + 2\beta$.
7. f'' is continuous if and only if $\alpha > 2 + 2\beta$.

Problem 71. Prove the following two variations of L'Hôspital's rule.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$, $g'(x) \neq 0$ for all $x \gg 1$, and the limit $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists. Show that the limit $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ also exists, and
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}\,.$$
2. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Suppose that for some $x_0 \in \{a, b\}$, $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \infty$, $g'(x) \neq 0$, and the limit $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$ exists. Show that the limit $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ also exists, and
$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}\,.$$
3. Find an example that the limit $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ exists but the limit $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$ does not exist.

Problem 72. Let $f : (a, b) \to \mathbb{R}$ be differentiable everywhere on (a, b) except perhaps at $x = x_0 \in (a, b)$, and $\lim\limits_{x\to x_0} f'(x)$ exists. Show that f is differentiable at x_0, and $\lim\limits_{x\to x_0} f'(x) = f'(x_0)$.

§4.8 Integration of Functions of One Variable

Problem 73. Let $f, g : [a, b] \to \mathbb{R}$, g continuous, $f \geqslant 0$ and f Riemann integrable. Show that

1. fg is Riemann integrable.

2. $\exists\, x_0 \in (a,b)$ such that

$$\int_a^b f(x)g(x)dx = g(x_0)\int_a^b f(x)dx\,.$$

Problem 74. Let $f:[a,b]\to\mathbb{R}$ be differentiable and assume that f' is Riemann integrable. Prove that $\displaystyle\int_a^b f'(x)dx = f(b)-f(a)$.

Hint: Use the Mean Value Theorem.

Problem 75. Suppose that $f:[a,b]\to\mathbb{R}$ is Riemann integrable, $m \leqslant f(x) \leqslant M$ for all $x\in[a,b]$, and $\varphi:[m,M]\to\mathbb{R}$ is continuous. Show that $\varphi\circ f$ is Riemann integrable on $[a,b]$.