Exercise Problems for Advanced Calculus

MA2045, National Central University, Fall Semester 2013

§0.1 Sets and Functions

Problem 1. Let S and T be given sets, $A \subseteq S$, $B \subseteq T$, and $f : S \to T$. Show that

1. $f(f^{-1}(B)) \subseteq B$, and $f(f^{-1}(B)) = B$ if $B \subseteq f(S)$.
2. $f^{-1}(f(A)) \supseteq A$, and $f^{-1}(f(A)) = A$ if $f : S \to T$ is one-to-one.

Problem 2. If $f : S \to T$ is a function from S into T, show that the following are equivalent; that is, show that each one of the following implies the other two.

a. f is one-to-one.

b. For every y in T, the set $f^{-1}(\{y\})$ contains at most one point.

c. $f(D_1 \cap D_2) = f(D_1) \cap f(D_2)$ for all subsets D_1 and D_2 of S.

§1.1 Ordered Fields and the Number Systems

Problem 3. Let $\mathcal{F}$ be an ordered field. Show that

1. $|x| \geq 0$ for every $x \in \mathcal{F}$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathcal{F}$.
4. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathcal{F}$.
5. $\big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathcal{F}$.

Problem 4. True or false. Provide a proof if the statement is true, and provide a counter-example if the statement is wrong. (若敘述為真則證明之，反之則必須給反例)

1. $(Q, <)$ is an ordered field.
2. $(Q, \geq)$ is an ordered field.

§1.2 Completeness and the Real Number System

Problem 5. Fix $y > 1$. Complete the following.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt{y}$ in class).
2. Show that $y^n - 1 > n(y - 1)$ for all $n \in \mathbb{N}$; thus $y - 1 > n(y^{1/n} - 1)$.
3. If $t > 1$ and $n > (y - 1)/(t - 1)$, then $y^{1/n} < t$.

4. Show that $\lim_{n\to\infty} y^{1/n} = 1$ as $n \to \infty$.

Problem 6. Let x_n be a monotone increasing sequence such that $x_{n+1} - x_n \le \frac{1}{n}$. Must x_n converge?

Problem 7. Let $\mathcal{F}$ be an ordered field in which every strictly monotone increasing sequence bounded above converges. Prove that $\mathcal{F}$ is complete.

Problem 8. Complete the following.

1. Let $x \ge 0$ be a real number such that for any $\varepsilon > 0$, $x \le \varepsilon$. Show that $x = 0$.

2. Let $S = (0,1)$. Show that for each $\varepsilon > 0$ there exists an $x \in S$ such that $x < \varepsilon$.

§1.3 Least Upper Bounds

Problem 9. Let A be a non-empty set of $\mathbb{R}$ which is bounded below. Define the set $-A$ by $-A \equiv \big\{ -x \in \mathbb{R} \,\big|\, x \in A \big\}$. Prove that

$$\inf A = -\sup(-A)\,.$$

Problem 10. Let A, B be non-empty subset of $\mathbb{R}$. Define $A + B = \{x + y \,|\, x \in A, y \in B\}$. Justify if the following statements are true or false by providing a proof for the true statement and giving a counter-example for the false ones.

1. $\sup(A + B) = \sup A + \sup B$.

2. $\inf(A + B) = \inf A + \inf B$.

3. $\sup(A \cap B) \le \min\{\sup A, \sup B\}$.

4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

5. $\sup(A \cup B) \ge \max\{\sup A, \sup B\}$.

6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Problem 11. Let $S \subseteq \mathbb{R}$ be bounded below and non-empty. Show that

$$\inf S = \sup \big\{ x \in \mathbb{R} \,\big|\, x \text{ is a lower bound for } S \big\}\,.$$

Problem 12. Fix $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that

   (a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.

   (b) if r, s in $\mathbb{Q}$, then $b^{r \cdot s} = (b^r)^s$.

2. For $x \in \mathbb{R}$, let $B(x) = \{b^t \in \mathbb{R} \,|\, t \in \mathbb{Q}, t \le x\}$. Show that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. Therefore, it makes sense to define $b^x = \sup B(x)$ for $x \in \mathbb{R}$. Show that the law of exponents (for real exponents) are also valid.

3. Let $y > 0$ be given. Using 4 of Problem 5 to show that if $u, v \in \mathbb{R}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n.

4. Let $y > 0$ be given, and A be the set of all w such that $b^w < y$. Show that $x = \sup A$ satisfies $b^x = y$.

5. Prove that if x_1, x_2 are two real numbers satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.

The number x satisfying $b^x = y$ is called the logarithm of y to the base b, and is denoted by $\log_b y$.

Problem 13. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$. For $k \in \mathbb{N}$, let $S_k = \{x_k, x_{k+1}, \cdots, x_{k+n}, \cdots\}$, and use $\sup\limits_{n\ge k} x_n$ to denote the number $\sup S_k$, and similarly use $\inf\limits_{n\ge k} x_n$ to denote $\inf S_k$.

1. Let $y_k = \sup\limits_{n\ge k} x_n$ and $z_k = \inf\limits_{n\ge k} x_n$. Show that the sequence $\{y_k\}_{k=1}^\infty$ is decreasing, and $\{z_k\}_{k=1}^\infty$ is increasing.

2. Show that $\{x_n\}_{n=1}^\infty$ is bounded above if $\lim\limits_{k\to\infty} y_k$ exists.

3. Show that $\{x_n\}_{n=1}^\infty$ is bounded below if $\lim\limits_{k\to\infty} z_k$ exists.

4. Show that if $\{x_n\}_{n=1}^\infty$ is bounded above, then $\lim\limits_{k\to\infty} y_k = \inf\limits_{k\ge 1} y_k$.

5. Show that if $\{x_n\}_{n=1}^\infty$ is bounded below, then $\lim\limits_{k\to\infty} z_k = \sup\limits_{k\ge 1} z_k$.

6. Let $x_n = (-1)^n\big(1 - \dfrac{1}{n}\big)$. Find $\lim\limits_{k\to\infty} \sup\limits_{n\ge k} x_n$ and $\lim\limits_{k\to\infty} \inf\limits_{n\ge k} x_n$.

Problem 14. Prove or disprove the following statement: let $A \subseteq \mathbb{R}$ satisfy

$$\sup\Big\{\sum_{b\in B} |b| \,\Big|\, B \text{ is a non-empty finite subsets of } A\Big\} < \infty\,.$$

Then $\{x \in A \,|\, x \neq 0\}$ is countable.

§1.4 Cauchy Sequences

Problem 15. Let $\{a_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be two sequences in $\mathbb{R}$, and define $S_k = \sum\limits_{n=1}^{k} a_n$ (so $\{S_k\}_{k=1}^\infty$ is also a sequence). Suppose that $|x_n - x_{n+1}| < a_n$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^\infty$ converges if $\{S_k\}_{k=1}^\infty$ converges.

Problem 16. Let $f : \mathbb{R} \to \mathbb{R}$ be a function so that $|f(x) - f(y)| \le \dfrac{|x-y|}{2}$. Pick an arbitrary $x_1 \in \mathbb{R}$, and define $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence.

Problem 17. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two Cauchy sequence in $\mathbb{R}$. Show that the sequence $\big\{|x_n - y_n|\big\}_{n=1}^{\infty}$ converges.

Problem 18. Prove the following unproven statements from lecture note.

1. $x_n \to x$ as $n \to \infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded and x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$.

2. $x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x.

Problem 19. True or false. Provide a proof if the statement is true, and provide a counter-example if the statement is wrong.

1. If a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ satisfies $x_{n+1} - \epsilon_n \le x_n$ for $n \in \mathbb{N}$, where $\sum\limits_{n=1}^{\infty} \epsilon_n$ is an absolute convergent series; that is, the partial sum $\sum\limits_{n=1}^{k} |\epsilon_n|$ converges as $k \to \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges.

2. Let $\pi : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto (such map π is called a rearrangement), and $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence. Then $\big\{x_{\pi(n)}\big\}_{n=1}^{\infty}$ is also convergent.

§1.5 Cluster Points; lim inf and lim sup

Problem 20. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$
\begin{aligned}
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\le \liminf_{n\to\infty}(x_n + y_n) \le \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
&\le \limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\,; \\
(\liminf_{n\to\infty} |x_n|)(\liminf_{n\to\infty} |y_n|) &\le \liminf_{n\to\infty} |x_n y_n| \le (\liminf_{n\to\infty} |x_n|)(\limsup_{n\to\infty} |y_n|) \\
&\le \limsup_{n\to\infty} |x_n y_n| \le (\limsup_{n\to\infty} |x_n|)(\limsup_{n\to\infty} |y_n|)\,.
\end{aligned}
$$

Give examples showing that the equalities are generally not true.

Problem 21. Prove that

$$
\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \le \liminf_{n\to\infty} \sqrt[n]{|x_n|} \le \limsup_{n\to\infty} \sqrt[n]{|x_n|} \le \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}\,.
$$

Give examples to show that the equalities are not true in general. Is it true that $\lim\limits_{n\to\infty} \sqrt[n]{|x_n|}$ exists implies that $\lim\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$ also exists?

Problem 22. Given the following sets consisting of elements of some sequence of real numbers. Find their sup and inf, and also the limsup and liminf of the sequence.

1. $\big\{\cos m \,\big|\, m = 0, 1, 2, \cdots\big\}$.

2. $\{(1+\frac{1}{m})\sin\frac{m\pi}{6}\,|\,m=1,2,\cdots\}$.

Hint: For 1, first show that for all irrational α, the set

$$S=\{x\in[0,1]\,\big|\,x=k\alpha \pmod 1 \text{ for some } k\in\mathbb{N}\}$$

is dense in $[0,1]$; that is, for all $y\in[0,1]$ and $\varepsilon>0$, there exists $x\in S\cap(y-\varepsilon,y+\varepsilon)$. Then choose $\alpha=\frac{1}{2\pi}$ to conclude that

$$T=\{x\in[0,2\pi]\,\big|\,x=k \pmod{2\pi} \text{ for some } k\in\mathbb{N}\}$$

is dense in $[0,2\pi]$. To prove that S is dense in $[0,1]$, you might want to consider the following set

$$S_k=\{x\in[0,1]\,\big|\,x=\ell\alpha \pmod 1 \text{ for some } 1\le\ell\le k+1\}$$

Note that there must be two points in S_k whose distance is less than $\frac{1}{k}$. What happened to (the multiples of) the difference of these two points?

Problem 23. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Show that

1. $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$.
2. If $\{x_n\}_{n=1}^{\infty}$ is bounded above by M, then $\limsup_{n\to\infty} x_n \leqslant M$.
3. If $\{x_n\}_{n=1}^{\infty}$ is bounded below by m, then $\limsup_{n\to\infty} x_n \geqslant m$.
4. $\limsup_{n\to\infty} x_n=\infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is not bounded above.
5. $\liminf_{n\to\infty} x_n=-\infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is not bounded below.
6. If x is a cluster point of $\{x_n\}_{n=1}^{\infty}$, then $\liminf_{n\to\infty} x_n \leqslant x \leqslant \limsup_{n\to\infty} x_n$.
7. If $a=\liminf_{n\to\infty} x_n$ is finite, then a is a cluster point.
8. If $b=\limsup_{n\to\infty} x_n$ is finite, then b is a cluster point.
9. If $\{x_n\}_{n=1}^{\infty}$ converges to x in $\mathbb{R}$ if and only if $\liminf_{n\to\infty} x_n=\limsup_{n\to\infty} x_n=x\in\mathbb{R}$.

§1.6 Euclidean Space

Problem 24. Show that the p-norm on Euclidean space $\mathbb{R}^n$ given by

$$\|x\|_p\equiv\begin{cases}\Big(\sum_{i=1}^{n}|x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1\le p<\infty\,,\\ \max\{|x_1|,\cdots,|x_n|\} & \text{if } p=\infty\,,\end{cases}\qquad x=(x_1,\cdots,x_n)$$

is indeed a norm.

§1.7 Norms, Inner Products, and Metrics

Problem 25. Let $\mathcal{M}$ be the collection of all $n \times m$ matrices with real entries. Define a function $\|\cdot\| : \mathcal{M} \to \mathbb{R}$ by

$$\|A\| = \sup_{\substack{x \in \mathbb{R}^m \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2},$$

here we recall that $\|\cdot\|_2$ is the 2-norm on Euclidean space given by

$$\|x\|_2 = \Big(\sum_{i=1}^k x_i^2\Big)^{1/2} \quad \text{if} \quad x = (x_1, \cdots, x_k) \in \mathbb{R}^k.$$

Show that

1. $\|A\| = \sup\limits_{\substack{x \in \mathbb{R}^m \\ \|x\|_2 = 1}} \|Ax\|_2 = \inf\big\{M \in \mathbb{R} \,\big|\, \|Ax\|_2 \leq M\|x\|_2 \;\forall\, x \in \mathbb{R}^m\big\}$.

2. $\|Ax\|_2 \leq \|A\|\|x\|_2$ for all $x \in \mathbb{R}^m$.

3. $\|\cdot\|$ defines a norm on $\mathcal{M}$.

§2.1 Open Sets

Problem 26. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $\mathcal{U} \subseteq \mathbb{R}$ is open, then

$$\mathcal{U} = \bigcup_{k \in \mathcal{I}} (a_k, b_k),$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \emptyset$ if $k \neq \ell$.

Problem 27. Let (M, d) be a metric space, and $A \subseteq M$. An open cover of A is a collection of open sets whose union contains A; that is, $\{\mathcal{U}_i\}_{i \in \mathcal{I}}$ is called an open cover of A if

1. $\mathcal{U}_i$ is open for all $i \in \mathcal{I}$.

2. $A \subseteq \bigcup\limits_{i \in \mathcal{I}} \mathcal{U}_i$.

Show that if $\big\{(a_k, b_k)\big\}_{k=1}^{\infty}$ is an open cover of $[a, b] \subseteq \mathbb{R}$, then there exists $N > 0$ such that $\bigcup\limits_{k=1}^{N} (a_k, b_k) \supseteq [a, b]$.

§2.2 Interior of a set

Problem 28. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{int}(\text{int}(A)) = \text{int}(A)$.

2. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

3. $\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$. Find examples of that $\text{int}(A \cup B) \supsetneq \text{int}(A) \cup \text{int}(B)$.

§2.4 Accumulation Points, Limit Points, and Isolated Points

Problem 29. True or false. Provide a proof if the statement is true, and provide a counter-example if the statement is wrong.

1. An interior point of a subset A of a metric space (M, d) is an accumulation point of that set.

2. Let (M, d) be a metric space, and $A \subseteq M$. Then $(A')' = A'$.

§2.3 Closed Sets, §2.5 Closure of Sets

Problem 30. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

3. $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$. Find examples of that $\text{cl}(A \cap B) \subsetneq \text{cl}(A) \cap \text{cl}(B)$.

§2.6 Boundary of Sets

Problem 31. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Show that

$$\partial A = \big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big).$$

Problem 32. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial A = \partial(M \backslash A)$.

2. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$.

3. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.

4. If $\text{cl}(A) \cap \text{cl}(B) = \emptyset$, then $\partial(A \cup B) = \partial A \cup \partial B$.

5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Problem 33. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Determine which of the following statements are true.

1. $\text{int}A = A \backslash \partial A$.

2. $\text{cl}(A) = M \backslash \text{int}(M \backslash A)$.

3. $\text{int}(\text{cl}(A)) = \text{int}(A)$.

4. $\text{cl}(\text{int}(A)) = A$.

5. $\partial(\mathrm{cl}(A)) = \partial A$.

6. If A is open, then $\partial A \subseteq M \backslash A$.

7. If A is open, then $A = \mathrm{cl}(A) \backslash \partial A$. How about if A is not open?

Problem 34. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be perfect if $A = A'$. The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the intervals

$$\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the Cantor set.

1. Show that C is a perfect set.

2. Show that C is uncountable.

3. Find $\mathrm{int}(C)$.

Problem 35. In a metric space (M, d), if subsets satisfy $A \subseteq S \subseteq \mathrm{cl}(A)$, then A is said to be dense in S. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.

2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \mathrm{cl}(A \cap B)$.

§2.7 Sequences, §2.8 Completeness

Problem 36. Show that

1. Every convergent sequence in a metric space is a Cauchy sequence.

2. If a subsequence of a Cauchy sequence converges to x, then the sequence converges to x.

3. x is a cluster point of $\{x_k\}_{k=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$ and $N > 0$, $\exists\, k > N$ with $d(x_k, x) < \varepsilon$.

4. x is a cluster point of $\{x_k\}_{k=1}^\infty$ if and only if there is a subsequence converging to x.

5. $x_k \to x$ as $k \to \infty$ if and only if every subsequence of $\{x_k\}_{k=1}^\infty$ converges to x.

6. $x_k \to x$ as $k \to \infty$ if and only if every proper subsequence of $\{x_k\}_{k=1}^\infty$ has a further subsequence that converges to x.

Problem 37. Let (M, d) be a metric space, and $N \subseteq M$. Show that if (N, d) is complete, then N is closed.

Remark: In class we have shown that if (M, d) is a complete metric space, and N is a closed subset of M, then (N, d) is complete. This problem gives a reverse statement.

Problem 38. Let (M, d) be a metric space. Call two Cauchy sequences $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ in M equivalent, denoted by $\{p_n\}_{n=1}^\infty \sim \{q_n\}_{n=1}^\infty$, if

$$\lim_{n\to\infty} d(p_n, q_n) = 0\,.$$

1. Prove that $\sim$ is an equivalence relation; that is, show that

   (a) $\{p_n\}_{n=1}^\infty \sim \{p_n\}_{n=1}^\infty$.

   (b) If $\{p_n\}_{n=1}^\infty \sim \{q_n\}_{n=1}^\infty$, then $\{q_n\}_{n=1}^\infty \sim \{p_n\}_{n=1}^\infty$.

   (c) If $\{p_n\}_{n=1}^\infty \sim \{q_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty \sim \{r_n\}_{n=1}^\infty$, then $\{p_n\}_{n=1}^\infty \sim \{r_n\}_{n=1}^\infty$.

2. Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be two Cauchy sequence, show that the sequence $\big\{d(p_n, q_n)\big\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$; thus is convergent.

3. Let M^* be the set of all equivalence classes. If $P, Q \in M^*$, we define

$$d^*(P, Q) = \lim_{n\to\infty} d(p_n, q_n)\,,$$

   where $\{p_n\}_{n=1}^\infty \in P$ and $\{q_n\}_{n=1}^\infty \in Q$. Show that the definition above is well-defined; that is, show that f $\{p'_n\}_{n=1}^\infty \in P$ and $\{q'_n\}_{n=1}^\infty \in Q$ are another two Cauchy sequences, then $\lim\limits_{n\to\infty} d(p_n, q_n) = \lim\limits_{n\to\infty} d(p'_n, q'_n)$.

4. Define $\varphi : M \to M^*$ as follows: for each $x \in M$, $\{x_n\}_{n=1}^\infty$, where $x_n \equiv x$ for all $n \in \mathbb{N}$, is a Cauchy sequence in M. Then $\{x_n\}_{n=1}^\infty \in \varphi(x)$ for one particular $\varphi(x) \in M^*$. In other words, $\varphi(x)$ is the equivalence class where $\{x_n\}_{n=1}^\infty$ belongs to. Show that

$$d^*\big(\varphi(x), \varphi(y)\big) = d(x, y) \qquad \forall\, x, y \in M.$$

5. Show that $\varphi(M)$ is dense in M^*.

6. Show that (M^*, d^*) is a complete metric space. The metric space (M^*, d^*) is called the completion of (M, d).

§2.9 Series of Real Numbers and Vectors

Problem 39. Let a_n be defined by $a_n = \begin{cases} \dfrac{n+1}{2^n} & \text{if } n \text{ is odd}\,, \\ \dfrac{n}{3^n} & \text{if } n \text{ is even}\,. \end{cases}$ Compute the value of $\liminf\limits_{n\to\infty} \sqrt[n]{a_n}$, $\limsup\limits_{n\to\infty} \sqrt[n]{a_n}$, $\liminf\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ and $\limsup\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$, and conclude that whether the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent or not.

Hint: You can use $\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} \sqrt[n]{n+1} = 1$ without proving it.

§3.1 Compactness

Problem 40. A metric space (M, d) is said to be separable if there is a countable subset A which is dense in M. Show that every compact set is separable.

Problem 41. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

1. Show that d is a metric on $\mathbb{R}^2$. In other words, $(\mathbb{R}^2, d)$ is a metric space.
2. Find $D(x, r)$ with $r < 1$, $r = 1$ and $r > 1$.
3. Show that the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is closed and bounded.
4. Examine whether the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is compact or not.

Problem 42. Let (M, d) be a complete metric space, and $A \subseteq M$ be totally bounded. Show that $\mathrm{cl}(A)$ is compact.

Problem 43. Let (M, d) be a metric space, and $K \subseteq M$. Show that K is compact if and only if for any family of closed subsets $\{F_\alpha\}_{\alpha \in I}$, we have

$$K \cap \bigcap_{\alpha \in I} F_\alpha \neq \emptyset$$

whenever

$$K \cap \bigcap_{\alpha \in J} F_\alpha \neq \emptyset \text{ for all } J \subseteq I \text{ satisfying } \#J < \infty.$$

Problem 44. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space, and $x_k \to x$ as $k \to \infty$. Show that the set $A \equiv \{x_1, x_2, \cdots,\} \cup \{x\}$ is compact by

1. showing that A is sequentially compact; and
2. showing that every open cover of A has a finite subcover; and
3. showing that A is totally bounded and complete.

Problem 45. Let (M, d) be a metric space.

1. Show that the union of a finite number of compact subsets of M is compact.

2. Show that the intersection of an arbitrary collection of compact subsets of M is compact.

Problem 46. Let X be the collection of all sequences $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that $\sup\limits_{k\geq 1} |x_k| < \infty$. In other words,

$$X = \left\{ \{x_k\}_{k=1}^{\infty} \,\middle|\, x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \sup_{k\geq 1} |x_k| < \infty \right\}.$$

Define $\|\cdot\| : X \to \mathbb{R}$ by

$$\left\| \{x_k\}_{k=1}^{\infty} \right\| = \sup_{k\geq 1} |x_k|.$$

1. Show that $\|\cdot\|$ is a norm on X. The normed space $(X, \|\cdot\|)$ usually is denoted by ℓ^{∞}.

2. Show that $(X, \|\cdot\|)$ is complete.

3. Let A, B, C, D be a subsets of X given by

$$\begin{aligned}
A &= \left\{ \{x_k\}_{k=1}^{\infty} \,\middle|\, |x_k| \leq \frac{1}{k} \text{ for all } k \in \mathbb{N} \right\}, \\
B &= \left\{ \{x_k\}_{k=1}^{\infty} \,\middle|\, x_k \to 0 \text{ as } k \to \infty \right\}, \\
C &= \left\{ \{x_k\}_{k=1}^{\infty} \,\middle|\, \text{the sequence } \{x_k\}_{k=1}^{\infty} \text{ converges} \right\}, \\
D &= \left\{ \{x_k\}_{k=1}^{\infty} \,\middle|\, \sup_{k\geq 1} |x_k| = 1 \right\}.
\end{aligned}$$

   Determine whether A, B, C, D are compact or not.

Problem 47. Let A, B be two non-empty subsets in $\mathbb{R}^n$. Define

$$d(A, B) = \inf \left\{ \|x - y\|_2 \,\middle|\, x \in A, y \in B \right\}$$

to be the distance between A and B. When $A = \{x\}$ is a point, we write $d(A, B)$ as $d(x, B)$.

(1) Prove that $d(A, B) = \inf \left\{ d(x, B) \,\middle|\, x \in A \right\}$.

(2) Show that $\left| d(x_1, B) - d(x_2, B) \right| \leq \|x_1 - x_2\|_2$ for all $x_1, x_2 \in \mathbb{R}^n$.

(3) Define $B_\varepsilon = \left\{ x \in \mathbb{R}^n \,\middle|\, d(x, B) < \varepsilon \right\}$ be the collection of all points whose distance from B is less than ε. Show that B_ε is open and $\bigcap\limits_{\varepsilon > 0} B_\varepsilon = \text{cl}(B)$.

(4) If A is compact, show that there exists $x \in A$ such that $d(A, B) = d(x, B)$.

(5) If A is closed and B is compact, show that there exists $x \in A$ and $y \in B$ such that $d(A, B) = d(x, y)$.

(6) If A and B are both closed, does the conclusion of (5) hold?

§3.2 The Heine-Borel Theorem

Problem 48. Let $M = \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\right\}$ with the standard metric $\|\cdot\|_2$. Show that $A \subseteq M$ is compact if and only if A is closed.

§3.3 Nested Set Property

Problem 49. 1. Let $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ be a sequence in $(\mathbb{R}, |\cdot|)$ that converges to x and let $A_k = \{x_k, x_{k+1}, \cdots\}$. Show that $\{x\} = \bigcap\limits_{k=1}^{\infty} \overline{A_k}$. Is this true in any metric space?

2. Suppose that $\{K_j\}_{j=1}^{\infty}$ is a sequence of comapct non-empty sets satisfying the nested set property; that is, $K_j \supseteq K_{j+1}$, and diameter$(K_j) \to 0$ as $j \to \infty$, where

$$\text{diameter}(K_j) = \sup\left\{d(x, y) \,\middle|\, x, y \in K_j\right\}.$$

Show that there is exactly one point in $\bigcap\limits_{j=1}^{\infty} K_j$.

§3.4 Connectedness

Problem 50. Let (M, d) be a metric space, and $A \subseteq M$. Show that A is disconnected (not connected) if and only if there exist non-empty closed set F_1 and F_2 such that

1. $A \cap F_1 \cap F_2 = \emptyset$; 2. $A \cap F_1 \neq \emptyset$; 3. $A \cap F_2 \neq \emptyset$; 4. $A \subseteq F_1 \cup F_2$.

Problem 51. Prove that if A is connected in a metric space (M, d) and $A \subseteq B \subseteq \overline{A}$, then B is connected.

Problem 52. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Suppose that A is connected and contain more than one point. Show that $A \subseteq A'$.

Problem 53. Show that the Cantor set C defined in Problem 34 is totally disconnected; that is, if $x, y \in C$, and $x \neq y$, then $x \in \mathcal{U}$ and $y \in \mathcal{V}$ for some open sets $\mathcal{U}$, $\mathcal{V}$ separate C.

Problem 54. Let F_k be a nest of connected compact sets (that is, $F_{k+1} \subseteq F_k$ and F_k is connected for all $k \in \mathbb{N}$). Show that $\bigcap\limits_{k=1}^{\infty} F_k$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that F_k is a nest of closed connected sets.

§4.1 Continuity

Started from this section, for all $n \in \mathbb{N}$ $\mathbb{R}^n$ always denotes the normed space $(\mathbb{R}^n, \|\cdot\|_2)$.

Problem 55. Complete the following.

1. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x, y)$$

exist but are not equal.

2. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits above exist and are equal but f is not continuous.

3. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous.

Problem 56. Complete the following.

1. Show that the projection map $f : \begin{array}{rl} \mathbb{R}^2 & \to \mathbb{R} \\ (x, y) & \mapsto x \end{array}$ is continuous.

2. Show that if $\mathcal{U} \subseteq \mathbb{R}$ is open, then $A = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x \in \mathcal{U}\right\}$ is open.

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $\mathcal{U} \subseteq \mathbb{R}$ such that $f(\mathcal{U})$ is not open.

Problem 57. Show that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is continuous if and only if for every $B \subseteq A$,

$$f(\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B)) .$$

Problem 58. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|$. Show that f is continuous on $(\mathbb{R}^n, \|\cdot\|_2)$.
Hint: Show that $|f(x) - f(y)| \le C\|x - y\|_2$ for some fixed constant $C > 0$.

§4.2, §4.3, §4.4, §4.5

Problem 59. Complete the following.

1. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $B \subseteq \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact, is $f^{-1}(K)$ necessarily compact?

3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $C \subseteq \mathbb{R}$ is connected, is $f^{-1}(C)$ necessarily connected?

Problem 60. Consider a compact set $K \subseteq \mathbb{R}^n$ and let $f : K \to \mathbb{R}^m$ be continuous and one-to-one. Show that the inverse function $f^{-1} : f(K) \to K$ is continuous. How about if K is not compact but connected?

§4.6 Uniform Continuity

Problem 61. Check if the following functions on uniformly continuous.

1. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin \log x$.

2. $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x \sin \dfrac{1}{x}$.

3. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.

Problem 62. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map.

1. Show that f is uniformly continuous on A if and only if for every pair of sequence $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ such that $d(x_k, y_k) \to 0$ as $k \to \infty$, then $\rho\big(f(x_k), f(y_k)\big) \to 0$ as $k \to \infty$.

2. Show that if f is uniformly continuous on A, and $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, then $\big\{f(x_k)\big\}_{k=1}^{\infty}$ is Cauchy.

Problem 63. Complete the following.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function; that is, $\exists\, p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$ (and f is continuous). Show that f is uniformly continuous on $\mathbb{R}$.

2. Suppose that $a, b \in \mathbb{R}$ and $f : (a,b) \to \mathbb{R}$ is continuous. Show that f is uniformly continuous on (a,b) if and only if the two limits

$$\lim_{x \to a^+} f(x) \quad \text{and} \quad \lim_{x \to b^-} f(x)$$

exist. How about if (a,b) is not a finite interval?

3. Suppose that $f : [a,b] \to \mathbb{R}$ is ***Hölder continuous with exponent*** α; that is, there exist $M > 0$ and $\alpha \in (0,1]$ such that

$$|f(x_1) - f(x_2)| \le M|x_1 - x_2|^{\alpha} \qquad \forall\, x_1, x_2 \in [a,b]\,.$$

Show that f is uniformly continuous on $[a,b]$. Show that $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\frac{1}{2}$.

Problem 64. Let (M,d) be a metric space, $A \subseteq M$, and $f, g : A \to \mathbb{R}$ be uniformly continuous on A. Show that if f and g are bounded, then fg is uniformly continuous on A. Does the conclusion still hold if f or g is not bounded?

§4.7 Differentiation of Functions of One Variable

Problem 65. Show that $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ if and only if there exists $m \in \mathbb{R}$, denoted by $f'(x_0)$, such that

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big|f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \le \varepsilon |x - x_0| \text{ if } |x - x_0| < \delta\,.$$

Problem 66. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, and $f \ge 0$. Find $\dfrac{d}{dx} f(x)^{g(x)}$.

Problem 67. Suppose α and β are real numbers, $\beta > 0$ and $f : [-1,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^{\alpha} \sin(x^{-\beta}) & \text{if } x \ne 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Prove the following statements.

1. f is continuous if and only if $\alpha > 0$.

2. $f'(0)$ exists if and only if $\alpha > 1$.

3. f' is bounded if and only if $\alpha \geq 1 + \beta$.

4. f' is continuous if and only if $\alpha > 1 + \beta$.

5. $f''(0)$ exists if and only if $\alpha > 2 + \beta$.

6. f'' is bounded if and only if $\alpha \geq 2 + 2\beta$.

7. f'' is continuous if and only if $\alpha > 2 + 2\beta$.

§4.8 Integration of Functions of One Variable

Problem 68. Let $f, g : [a, b] \to \mathbb{R}$, g continuous, $f \geq 0$ and f Riemann integrable. Show that

1. fg is Riemann integrable.

2. $\exists\, x_0 \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = g(x_0) \int_a^b f(x)dx\,.$$

Problem 69. Let $f : [a, b] \to \mathbb{R}$ be differentiable and assume that f' is Riemann integrable. Prove that $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$.

Hint: Use the Mean Value Theorem.

Problem 70. Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable, $m \leq f(x) \leq M$ for all $x \in [a, b]$, and $\varphi : [m, M] \to \mathbb{R}$ is continuous. Show that $\varphi \circ f$ is Riemann integrable on $[a, b]$.

Problem 71. Let $A \subseteq \mathbb{R}$ be a bounded set, and $f : A \to \mathbb{R}$ be bounded. Then f is said to be integrable on A with integral I if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0 \ni$ if $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is any partition of A with mesh size less than δ and $\{\xi_1, \cdots, \xi_n\}$ is any collection of points with the property that $\xi_k \in [x_{k-1}, x_k]$ for all k, then

$$\Big|\sum_{k=1}^{n} f(\xi_k)(x_{k+1} - x_k) - I\Big| < \varepsilon.$$

The number $\sum\limits_{k=1}^{n} f(\xi_k)(x_{k+1} - x_k)$ is called a Riemann sum. Show that f is Riemann integrable on A if and only if f is integrable on A.