§ 5.8

#4) A polynomial of two variables (i.e. on $[0,1]\times[0,1]$) takes the form

$$(*)\quad P(x,y)=\sum_{j=0}^{N}\sum_{i=0}^{K} a_{ij}\,x^i y^j \qquad \text{with degree } K+N$$

We will show that $\mathcal{P}=\{$ polynonomials in two variables $\}$ satisfies the assumptions in 5.8.2

(i) $\mathcal{P}$ is an algebra

— If $P,q\in\mathcal{P}\Rightarrow \lambda P,\ P+q,\ Pq\in\mathcal{P}$. Clearly, if P,q takes the form $(*)$, $P+q$, Pq, λP all take the form $(*)$.

(ii) $P(x,y)\equiv 1$ is of the form $(*)$ (take $N=K=0$ $a_{00}=1$)

(iii) Given $(x_0,y_0)\neq(x_1,y_1)$ then either $x_0\neq x_1$ and $y_0\neq y_1$ or $x_0\neq x_1$, $y_0=y_1$. in this case take $P(x,y)=x$. Otherwise $y_0\neq y_1$ take $P(x,y)=y$. then $P(x_0,y_0)\neq P(x_1,y_1)$

Hence 5.8.2 $\Rightarrow$ $\mathcal{P}$ is dense in $(\mathcal{C}([0,1]\times[0,1];\mathbb{R}),\ \|\ \|_\infty)$

#6) If $(x,y)\in T$ then $\exists\,\theta\in[0,2\pi]$ s.t. $(x,y)=(\cos\theta,\sin\theta)$ and for each $\theta\in[0,2\pi]$ $(\cos\theta,\sin\theta)\in T$.

$$\theta\in[0,2\pi]\longleftrightarrow \underset{\in T}{(\cos\theta,\sin\theta)}\rightarrow f(\theta)$$

We show $\mathcal{T}=\left\{\sum_{k=1}^{N}\left(a_k\cos(k\theta)+b_k\sin(k\theta)\right)+a_0/2\ ,\ \theta\in[0,2\pi],\ a_k,b_k\in\mathbb{R},\ N\in\mathbb{N}\right\}$ is dense in $(\mathcal{C}(T;\mathbb{R}),\ \|\cdot\|_\infty)$.

Again it suffices to show $\mathcal{T}$ satisfies the assumptions in 5.8.2.

(i) $\mathcal{T}$ is an algebra. The only property that is not easy to see is $f,g\in\mathcal{T}$, $fg\in\mathcal{T}$.

$$f=\sum_{k=1}^{N}\left(a_k\cos(k\theta)+b_k\sin(k\theta)\right)+a_0/2 \qquad g=\sum_{j=1}^{K}\left(\tilde a_j\cos(j\theta)+\tilde b_j\sin(j\theta)\right)+\tilde a_0/2$$

→ In the above, only the main points. You need to work out the details

Chapter 5.

#11

a. Must a contraction on any metric space have a fixed point?

Ans: (i) If the metric space is not complete, then a contraction mapping may not have a fixed point.

example: Let $S = Q^c \cap [0,1] \Rightarrow S$ is not complete.

Let $f(x) = 1 - \frac{x}{2}, \ \forall x \in S$

$|f(x) - f(y)| = \frac{1}{2}|x-y|, \ \forall x, y \in S$

$\Rightarrow f$ is a contraction mapping on S.

Suppose $x_0 \in S$ is the fixed point of f

i.e. $\underset{1-\frac{x_0}{2}}{\underset{\parallel}{f(x_0)}} = x_0 \Rightarrow \frac{3}{2}x_0 = 1$

$\Rightarrow x_0 = \frac{2}{3} \notin S \rightarrow\leftarrow$

$\Rightarrow f$ has no fixed point on S. #

(ii) If S is complete and $d(f(x), f(y)) < k\,d(x,y)$, $k \geq 1$, $\forall x, y \in S$, then f may not have a fixed point.

example: Let $S = R \Rightarrow S$ is complete.

Let $f(x) = 1 + x \ \forall x \in S$.

$\Rightarrow |f(x) - f(y)| = |x - y|$

f has no fixed point #.

Chapter 5.

#11.

b. Let $f: X \to X$, where X is a complete metric space.

Satisfy $d(f(x), f(y)) < d(x,y)$ $\forall x, y \in X$ s.t. $x \neq y$.

Must f have a fixed point? What if X is compact?

Ans: f may not have a fixed point.

Let $X = [0, \infty)$. X is a complete metric space.

Let $f: X \to X$ be defined by

$$f(x) = -1 + x - \log(1+x).$$

Then for $0 < x < y$

$$0 < f(y) - f(x) = (y-x) - \left(\log(1+y) - \log(1+x)\right)$$

$$= (y-x) - \frac{1}{1+z}(y-x), \quad z \in (x,y).$$

$$= (y-x)\left(\frac{z}{1+z}\right)$$

$$< (y-x)$$

But f has no fixed point in $[0, \infty)$.

If X is compact, f has a unique fixed point.

(the details see section 5.8 #8(a).) ※

補充 Section 5.8 #8(a).

The fixed point of Φ is unique.

pf: Suppose $\exists$ x_1, x_2 are fixed points of Φ

i.e. $\Phi(x_1) = x_1$, $\Phi(x_2) = x_2$, $x_1 \neq x_2$ $(d(x_1, x_2) > 0)$

$\Rightarrow d(\Phi(x_1), \Phi(x_2)) < d(x_1, x_2)$ $\to\leftarrow$ $\therefore \Phi$ has only one fixed point.

Chapter 5.

#12.

a. Show that f is continuous iff it is both upper and lower semicontinuous.

Pf: "$\Rightarrow$" Let f be continuous and $x_0 \in A$.

Then for any $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|f(x)-f(x_0)| < \varepsilon \quad \forall\, x \in D(x_0, \delta) \cap A$$

$$\Rightarrow \quad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$$

Now suppose that $\lambda < f(x_0)$

Let $\varepsilon = f(x_0) - \lambda > 0$

Then $f(x_0) - \varepsilon = \lambda < f(x) \quad \forall\, x \in D(x_0, \delta) \cap A$

$\Rightarrow$ f is lower semicontinuous at x_0.

Let $\lambda' > f(x_0)$ and $\varepsilon = \lambda' - f(x_0) > 0$.

Then $f(x) < f(x_0) + \varepsilon = \lambda' \quad \forall\, x \in D(x_0, \delta) \cap A$

$\Rightarrow$ f is upper semicontinuous at x_0. #

"$\Leftarrow$" Suppose f is both upper and lower semicontinuous

Let $\varepsilon > 0$ and $x_0 \in A$ be given.

Then let $\lambda = f(x_0) - \varepsilon$, we find $\delta_1 > 0$ s.t. $\forall\, x \in D(x_0, \delta_1) \cap A$

$$f(x_0) - \varepsilon = \lambda < f(x).$$

Let $\lambda' = f(x_0) + \varepsilon$, we find $\delta_2 > 0$ s.t. $\forall\, x \in D(x_0, \delta_2) \cap A$

$$f(x) < \lambda' = f(x_0) + \varepsilon.$$

Now let $\delta = \min\{\delta_1, \delta_2\}$, then $\forall\, x \in D(x_0, \delta) \cap A$

we get $\lambda = f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon = \lambda'$

$\Rightarrow$ f is continuous at x_0

~~Since~~ $x_0 \in A$ is arbitrary, f is continuous #

b. If the functions f_K are lower semicontinuous, $f_K \to f$ pointwise, and $f_{K+1}(x) \geq f_K(x)$, then prove that f is lower semicontinuous.

Pf: Let $x_0 \in A$ and $\lambda < f(x_0)$.

Let $\varepsilon = f(x_0) - \lambda > 0$.

Since $f_K \nearrow f$, we can choose $N \in \mathbb{N}$ s.t.

$$f(x_0) - \frac{\varepsilon}{2} < f_N(x_0).$$

Now since f_N is lower semicontinuous at x_0, we have $\delta > 0$ s.t. $\forall\ x \in D(x_0, \delta) \cap A$

$$f_N(x_0) - \frac{\varepsilon}{2} < f_N(x).$$

Hence if $x \in D(x_0, \delta) \cap A$, we have

$$\lambda = f(x_0) - \varepsilon = \underline{f(x_0) - \frac{\varepsilon}{2}} - \frac{\varepsilon}{2} < f_N(x_0) - \frac{\varepsilon}{2} < f_N(x) \leq f(x)$$

$\Rightarrow$ f is lower semicontinuous ※

c. In b, show that f need not be continuous even if the f_K are continuous.

Pf: If $f_K \nearrow f$ is not uniformly;

Then f need not be continuous even if the f_K are continuous. ※

#12.

d. Let $f : [0,1] \to \mathbb{R}$, and let $g(x) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(y)$.

Prove that g is lower semicontinuous.

pf. Note: $\sup_{\delta>0} \inf_{|x-y|<\delta} f(y) = \lim_{\delta\to0^+} \left(\inf_{|x-y|<\delta} f(y) \right)$

Let $x_0 \in [0,1]$, $\lambda < g(x_0)$ be given.

Let $\varepsilon = g(x_0) - \lambda > 0$.

Since $g(x_0) = \lim_{\delta\to0^+} \left(\inf_{|y-x_0|<\delta} f(y) \right)$.

$\exists\ \delta_0 > 0$ s.t.

$$g(x_0) - \varepsilon < \inf_{|y-x_0|<\delta} f(y) \quad \forall\ \delta \le \delta_0$$

Suppose that $x \in D(x_0, \frac{\delta_0}{2})$.

Then if $y \in D(x, \frac{\delta_0}{2}) \Rightarrow y \in D(x_0, \delta_0)$

$$\Rightarrow g(x_0) - \varepsilon < \inf_{|y-x_0|<\delta_0} f(y) \le \inf_{|y-x|<\frac{\delta_0}{2}} f(y) \le g(x)$$

$\forall\ x \in D(x_0, \frac{\delta_0}{2})$.

$\Rightarrow g$ is lower semicontinuous at x_0.

Since x_0 is arbitrary, g is lower semicontinuous on $[0,1]$ ※

Chapter 5

21.

a. Prove that if $A \subset \mathbb{R}^n$ is compact, $\mathcal{B} \subset C(A, \mathbb{R}^m)$ is compact

$\Leftrightarrow$ $\mathcal{B}$ is closed, bounded, and equicontinuous.

Pf: "$\Rightarrow$" Since $\mathcal{B}$ is compact, $\mathcal{B}$ is closed and bounded.

Also, by Arzela-Ascoli Thm, $\mathcal{B}$ is equicontinuous. #

"$\Leftarrow$" Let $\mathcal{B}_{x_0} = \{ f(x_0) \mid f \in \mathcal{B} \}$ for fixed $x_0 \in A$.

Claim: $\mathcal{B}_{x_0}$ is compact in $\mathbb{R}^m$.

Pf: Since $\mathcal{B}$ is bounded.

i.e. $\exists M$ s.t. $\|f\| \leq M$, $\forall f \in \mathcal{B}$.

$\Rightarrow$ $|f(x_0)| \leq \|f\| \leq M$ $\forall f \in \mathcal{B}$.

$\Rightarrow$ $\mathcal{B}_{x_0}$ is bounded in $\mathbb{R}^m$

Let y_0 be the accumulation point of $\mathcal{B}_{x_0}$.

$\Rightarrow \exists \{x_k\} \in \mathcal{B}_{x_0}$ s.t. $x_k \to y_0$.

i.e. $\exists f_k \in \mathcal{B}$ s.t. $x_k = f_k(x_0) \to y_0$

Since $\mathcal{B}$ is equicontinuous and pointwise bdd.

by Corollary 5.6.3, $\exists$ convergent subsequence $\{f_{k_j}\}$ of $\{f_k\}$.

i.e. $f_{k_j} \to f$ in $C(A, \mathbb{R}^m)$, $f \in \mathcal{B}$ ($\because \mathcal{B}$ is closed)

$$\Rightarrow \underbrace{f_{k_j}(x_0)}_{= x_{k_j}} \to f(x_0)$$

$\Rightarrow$ $x_{k_j} \to y_0 = f(x_0)$ ($\because x_k \to y_0$)

$\Rightarrow$ $y_0 \in \mathcal{B}_{x_0} \Rightarrow \mathcal{B}_{x_0}$ is closed in $\mathbb{R}^m$

Thus $\mathcal{B}_{x_0}$ is compact in $\mathbb{R}^m$

Chapter 5

\# 21.

a.

"$\Leftarrow$" Since $\mathcal{B}$ is closed, equicontinuous and pointwise compact,

by Arzela-Ascoli Thm (5.6.2)

we get $\mathcal{B}$ is compact ※

\# 21

b.

Let $D = \{ f \in C([0,1], \mathbb{R}) \mid \|f\| \le 1 \}$.

Show that D is closed and bounded, but is not compact.

Pf: Let f be the accumulation point of D

$\Rightarrow \exists \{f_k\} \subset D$ s.t. $f_k \to f$ in $C([0,1], \mathbb{R})$

$\Rightarrow f_k(x) \to f(x)$ uniformly on $[0,1]$

$\Rightarrow$ Since $|f_k(x)| \le 1 \quad \forall k, \forall x \in [0,1]$

$\Rightarrow \quad |f(x)| = \lim_{k\to\infty} |f_k(x)| \le 1, \forall x \in [0,1]$

$\Rightarrow \|f\| \le 1 \Rightarrow f \in D$

$\Rightarrow D$ is closed.

Since $D = \{ f \in C[0,1], \mathbb{R}) \mid \|f\| \le 1 \} \Rightarrow D$ is bounded.

Claim: D is not compact.

Pf: Let $f_n(x) = x^n$, $x \in [0,1]$.

$\Rightarrow |f_n(x)| \le 1 \quad \forall x \in [0,1], \forall n$.

$\Rightarrow \|f_n\| \le 1 \quad \forall n$.

Suppose $\{f_n\}$ is equicontinuous.

Let $\varepsilon = \frac{1}{2}$, then $\exists \delta > 0$ s.t. $|1-y| < \delta$, $y \in [0,1)$

$\Rightarrow |f_n(1) - f_n(y)| = |1 - y^n| < \frac{1}{2}$, $\forall n$.

Since $y < 1$, $\exists N$ s.t. $y^N < \frac{1}{2} \Rightarrow |1 - y^N| > \frac{1}{2}$ $\rightarrow\leftarrow$

$\Rightarrow \{f_n\}$ is not equicontinuous

By a, D is not compact ※

Chapter 5.

#45.

a. Let $f_k : K \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be an equicontinuous sequence of functions on a compact set K converging pointwise. Prove that the convergence is uniform.

Pf: Given $\varepsilon > 0$, choose $\delta > 0$ s.t.

$$\| f_k(x) - f_k(y) \| < \frac{\varepsilon}{3} \text{ as } \|x - y\| < \delta \quad \forall k$$

$\bigcup_{x \in K} D(x, \delta)$ is an open cover of K.

Since K is compact, there exist $x_1, x_2, \dots, x_N$ such that $K \subset \bigcup_{i=1}^{N} D(x_i, \delta)$

Now choose $M \in \mathbb{N}$ s.t.

$$\| f_n(x_i) - f_m(x_i) \| < \frac{\varepsilon}{3} \quad \forall n, m \geq M \text{ and } \forall i = 1, 2, \dots, N.$$

Then if $n, m \geq M$, we have

$$\| f_n(x) - f_m(x) \| \leq \| f_n(x) - f_n(x_i) \| + \| f_n(x_i) - f_m(x_i) \| + \| f_m(x_i) - f_m(x) \|$$

where $x \in D(x_i, \frac{\delta}{2})$.

Thus $\| f_n(x) - f_m(x) \| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall x \in K$ as $n, m \geq M$.

$\Rightarrow \{f_n\}$ satisfies the Cauchy condition for uniformly convergence

$\Rightarrow \{f_n\}$ converges uniformly ※.

Chapter 5.

#45.

b. Let

$$f_n(x) = \frac{x^2}{x^2+(1-nx)^2}, \quad 0 \le x \le 1$$

Show that f_n converges pointwise but not uniformly.
What can you conclude from a ?

Pf: $f_n(x) \longrightarrow 0$ pointwise.

But $f_n'(x) = 0 \Rightarrow x = \frac{1}{n}$

$$f_n\left(\frac{1}{n}\right) = \frac{\left(\frac{1}{n}\right)^2}{\left(\frac{1}{n}\right)^2} = 1 \quad \forall n.$$

$\Rightarrow f_n \not\to 0$ uniformly.

From a, we get $\{f_n\}$ is not equicontinuous. #

#55.

See Section 5.3.

Example 5.3.9 (a.), (b.), (c.) #

50 Let P_n are polynomials and $P_n \to f$ uniformly

50. Let $P_n = a_n^m x^m + \cdots + a_n^0$

Check: $P_i - a_i^0 = P_j - a_j^0$ $\forall i, j \geq N$ and N large enough

$\because$ $P_n \to f$ uniformly

$\Rightarrow$ $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ $\ni$ $|P_n(x) - P_m(x)| < \varepsilon$ $n, m \geq N$ $\forall x \in \mathbb{R}$

Let $P_n = a_n^j x^j + \cdots + a_n^0$ $a_n^j \neq 0$

$P_m = a_m^k x^k + \cdots + a_m^0$ $a_m^k \neq 0$

If $j > k$ $|P_n(x) - P_m(x)| = |a_n^j x^j + \cdots + (a_n^k - a_m^k) x^k + \cdots + (a_n^0 - a_m^0)|$

$= |x^j (a_n^j + \cdots + (a_n^k - a_m^k) \frac{x^k}{x^j} + \cdots + \frac{(a_n^0 - a_m^0)}{x^j})|$

$= |x^j| \, |a_n^j + \cdots + \underbrace{(a_n^k - a_m^k)\frac{x^k}{x^j}}_{\to 0} + \cdots + \underbrace{\frac{(a_n^0 - a_m^0)}{x^j}}_{\to 0}|$

$\to \infty$ as $|x| \to \infty$ $\times$

$k > j$ similar

when $k = j$ check $a_n^j = a_m^j$

If $a_n^j \neq a_m^j$ $|P_n(x) - P_m(x)| = |(a_n^j - a_m^j) x^j + \cdots + (a_n^0 - a_m^0)|$

$= |x^j| \, |a_n^j - a_m^j + \underbrace{(a_n^{j-1} - a_m^{j-1}) \frac{1}{x}}_{\to 0} + \cdots \underbrace{\frac{(a_n^0 - a_m^0)}{x^j}}_{\to 0}|$

$\to \infty$ as $|x| \to \infty$ $\times$

Similar $a_n^i = a_m^i$ $0 < i \leq j$

$\Rightarrow$ f is a polynomial

57.

Check: $\left| \int_0^\infty f_n(x)\,dx - \int_0^\infty f(x)\,dx \right| \to 0$ as $n \to \infty$

pf $\forall \varepsilon > 0$

$\because \int_0^\infty g(x)\,dx < \infty \Rightarrow \exists R > 0 \ni \int_R^\infty g(x)\,dx < \frac{\varepsilon}{3} \Rightarrow \int_R^\infty |f_n|\,dx < \int_R^\infty g(x)\,dx < \frac{\varepsilon}{3} \quad \forall n$

$\because f_n \to f(x)$ uniformly $\Rightarrow |f(x)| \leq g(x) \Rightarrow \int_R^\infty |f(x)|dx < \frac{\varepsilon}{3}$

$$\left| \int_0^\infty f_n(x)\,dx - \int_0^\infty f(x)\,dx \right| \leq \left| \int_0^R f_n(x) - f(x)\,dx \right| + \left| \int_R^\infty f_n(x)\,dx \right| + \left| \int_R^\infty f(x)\,dx \right|$$

$\because f_n(x) \to f(x)$ uniformly $0 \leq x \leq R$

$\Rightarrow \exists N \in \mathbb{N} \ni \left| \int_0^R f_n(x) - f(x)\,dx \right| < \frac{\varepsilon}{3} \quad n \geq N$

$\Rightarrow \left| \int_0^\infty f_n(x)\,dx - \int_0^\infty f(x)\,dx \right| \leq \varepsilon$

$\Rightarrow \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx$

59 (a) Let $f(x)=\frac{x^p}{p}+\frac{x^{-q}}{q}$ $x\in(0,\infty)$

$\Rightarrow$ f is continuous and differentiable

$$f'(x)=x^{p-1}-x^{-q-1} \quad \begin{cases} <0 & \text{when } x\in(0,1) \\ =0 & \text{when } x=1 \\ >0 & \text{when } x\in(1,\infty) \end{cases}$$

$\Rightarrow f(1)=1\le\frac{x^p}{p}+\frac{x^{-q}}{q}$ for all $x>0$

Let $x=\frac{m^{1/q}}{n^{1/p}}$

$$\Rightarrow 1\le\frac{\frac{m^{p/q}}{n}}{p}+\frac{\frac{n^{q/p}}{m}}{q}$$

$$\frac{p}{q}+1=\frac{p+q}{q}=p \qquad \frac{1}{p}+\frac{1}{q}=1$$

$$\frac{q}{p}+1=\frac{p+q}{p}=q \qquad \frac{p+q}{pq}=1$$

$$\Rightarrow mn\le\frac{m^{p/q+1}}{p}+\frac{n^{q/p+1}}{q}$$

$$\Rightarrow mn\le\frac{m^p}{p}+\frac{n^q}{q}$$

Let $m=at$ $n=bt^{-1}$

$$\Rightarrow ab\le\frac{a^pt^p}{p}+\frac{b^qt^{-q}}{q}$$

(b) By (a) when $t=1$

$$\sum_1^n\frac{a_k}{(\sum_1^n a_k^p)^{1/p}}\frac{b_k}{(\sum_1^n b_k^q)^{1/q}}\le\sum_1^n\frac{1}{p}\frac{a_k^p}{(\sum_1^n a_k^p)}+\frac{1}{q}\frac{b_k^q}{(\sum_1^n b_k^q)}=\frac{1}{p}+\frac{1}{q}=1$$

$$\Rightarrow \sum_1^n a_kb_n\le(\sum_1^n a_k^p)^{\frac{1}{p}}(\sum_1^n b_k^q)^{\frac{1}{q}}$$

(c) By (b)

$$\sum_1^n(a_k+b_k)^p=\sum_1^n(a_k+b_k)(a_k+b_k)^{p-1}=\sum_1^n a_k(a_k+b_k)^{p-1}+b_k(a_k+b_k)^{p-1}$$

$$=\sum_1^n a_k(a_k+b_k)^{p-1}+\sum_1^n b_k(a_k+b_k)^{p-1}$$

$$\le(\sum_1^n a_k^p)^{\frac{1}{p}}(\sum_1^n(a_k+b_k)^{(p-1)q})^{\frac{1}{q}}+(\sum_1^n b_k^p)^{\frac{1}{p}}(\sum_1^n(a_k+b_k)^{(p-1)q})^{\frac{1}{q}}$$

$$=\left((\sum_1^n a_k^p)^{\frac{1}{p}}+(\sum_1^n b_k^p)^{\frac{1}{p}}\right)\left(\sum_1^n(a_k+b_k)^p\right)^{\frac{1}{q}})$$

59.

$$\Rightarrow \frac{\sum_1^n (a_k+b_k)^p}{\left(\sum_1^n (a_k+b_k)^p\right)^{\frac{1}{q}}} \le \left(\sum_1^n a_k^p\right)^{\frac{1}{p}} + \left(\sum_1^n b_k^p\right)^{\frac{1}{p}}$$

$\frac{1}{p}+\frac{1}{q}=1 \quad 1-\frac{1}{q}=\frac{1}{p}$

$$\Rightarrow \left(\sum_1^n (a_k+b_k)^p\right)^{1-\frac{1}{q}} = \left(\sum_1^n (a_k+b_k)^p\right)^{\frac{1}{p}} \le \left(\sum_1^n a_k^p\right)^{\frac{1}{p}} + \left(\sum_1^n b_k^p\right)^{\frac{1}{p}}$$

68 (a) Let $(f_n, x_n), (f, x) \in C(A, \mathbb{R}^m) \times A$

$(f_n, x_n) \to (f, x)$ as $n \to \infty$

$\Rightarrow f_n \to f$ in $C(A, \mathbb{R}^m)$ and $x_n \to x$ in $\mathbb{R}^n$

Check: $E((f_n, x_n)) \to E((f, x))$

Pf: $\|f_n(x_n) - f(x)\| \le \|f_n(x_n) - f(x_n)\| + \|f(x_n) - f(x)\|$

$\le \|f_n - f\| + \|f(x_n) - f(x)\|$

$\to 0$ as $n \to \infty$ ($\because f_n \to f$ and f is continuous $\Rightarrow f(x_n) \to f(x)$ as $x_n \to x$)

$\Rightarrow E$ is continuous

(b) A is compact and B is compact $\Rightarrow B \times A$ is compact

By 46.2 E is uniformly on $B \times A$

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \ni (f, x), (g, y) \in B \times A, \ \|(f, x) - (g, y)\| < \delta \quad \|E((f, x)) - E((g, y))\| = \|f(x) - g(y)\| < \varepsilon$

$\Rightarrow$ for any $(f, x), (f, y) \in B \times A$

$\|(f, x) - (f, y)\| = \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$

$\Rightarrow B$ is equicontinuous