§ 5.7

#4.

Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f = (f_1, f_2, f_3)$

$$\begin{cases} f_1(x_1, x_2, x_3) = \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{2}{15}x_3 + 3 \\ f_2(x_1, x_2, x_3) = \frac{1}{4}x_1 + \frac{1}{5}x_2 + \frac{1}{2}x_3 - 1 \\ f_3(x_1, x_2, x_3) = \frac{-1}{4}x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3 + 2 \end{cases}$$

Then $\|f(x) - f(y)\| = \|f(x_1, x_2, x_3) - f(y_1, y_2, y_3)\|$

$$= [(f_1(x) - f_1(y))^2 + (f_2(x) - f_2(y))^2 + (f_3(x) - f_3(y))^2]^{\frac{1}{2}}$$

Since $f_1(x_1, x_2, x_3) - f_1(y_1, y_2, y_3)$

$$= \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{2}{15}x_3 + \cancel{3} - \frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{2}{15}y_3 - \cancel{3}$$

$$= \frac{1}{4}(x_1 - y_1) - \frac{1}{4}(x_2 - y_2) + \frac{2}{15}(x_3 - y_3)$$

$$= \left(\frac{1}{4}, -\frac{1}{4}, \frac{2}{15}\right) \cdot (x_1 - y_1, x_2 - y_2, x_3 - y_3)$$

$$\Rightarrow \quad |f_1(x_1, x_2, x_3) - f_1(y_1, y_2, y_3)| = \left|\left(\frac{1}{4}, -\frac{1}{4}, \frac{2}{15}\right) \cdot (x_1 - y_1, x_2 - y_2, x_3 - y_3)\right|$$

$$\le \left[\left(\frac{1}{4}\right)^2 + \left(\frac{-1}{4}\right)^2 + \left(\frac{2}{15}\right)^2\right]^{\frac{1}{2}} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right]^{\frac{1}{2}}$$

Thus $(f_1(x) - f_1(y))^2 \le \left[\left(\frac{1}{4}\right)^2 + \left(\frac{-1}{4}\right)^2 + \left(\frac{2}{15}\right)^2\right]\left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right]$ $\quad (x_1, x_2, x_3) - (y_1, y_2, y_3)$

$= \left[\left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(\frac{2}{15}\right)^2\right] \|x - y\|^2$ $\quad (x_1 - y_1, x_2 - y_2, x_3 - y_3)$

Similarly, $(f_2(x) - f_2(y))^2 \le \left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{4}\right)^2\right] \|x - y\|^2$

$$(f_3(x) - f_3(y))^2 \le \left[\left(\frac{-1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] \|x - y\|^2$$

Therefore, $\|f(x) - f(y)\| \le \left[\left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(\frac{2}{15}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{-1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^{\frac{1}{2}} \|x - y\|$

$$\le M\|x - y\|, \quad M < 1$$

$\Rightarrow$ $f: \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction mapping. Hence f has a unique fixed point. Thus the equation has a unique solution.

#8

a. Let $g(x) = d(x, \Phi(x))$, $\forall x \in M$.

$\because$ $d(x_1, \Phi(x_1)) \leq d(x_1, x_2) + d(x_2, \Phi(x_2)) + d(\Phi(x_2), \Phi(x_1))$

$\therefore$ $g(x_1) - g(x_2) \leq d(x_1, x_2) + d(\Phi(x_1), \Phi(x_2))$

$< 2\, d(x_1, x_2)$

Change x_1, x_2, we get

$g(x_2) - g(x_1) < 2d(x_2, x_1)$

Thus, $|g(x_1) - g(x_2)| < 2d(x_1, x_2)$, $\forall x_1, x_2 \in M$

That is, g is uniformly conti on M

$\because$ g is conti on cpt set M

$\therefore$ $\exists\, x_0$ s.t. $g(x_0)$ is the minimum on M

i.e. $0 \leq g(x_0) \leq g(x)$, $\forall x \in M$.

If $g(x_0) = d(x_0, \Phi(x_0)) \neq 0$, i.e. $\Phi(x_0) \neq x_0$,

then $g(\Phi(x_0)) = d(\Phi(x_0), \Phi^2(x_0)) < d(x_0, \Phi(x_0)) = g(x_0)$ $\rightarrow\leftarrow$

$\Rightarrow$ $g(x_0) = 0$ $\Rightarrow$ $\Phi(x_0) = x_0$.

Thus, Φ has a unique fixed point.

b. Let $M = (0, \frac{1}{2})$, $\Phi(x) = x^2$, $\forall x \in M$,

then $|\Phi(x) - \Phi(y)| = |x+y||x-y| < |x-y|$, $\forall x, y \in M$

But Φ has no fixed point on M. #

#26. Let $T: C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$

by $(Tf)(x) = A(x) + \int_0^1 f(y)K(x,y)\,dy$

Claim:

T is a contraction mapping

Pf: For $f, g \in C([0,1],\mathbb{R})$

$$\|T(f)-T(g)\| = \sup_{x\in[0,1]} |T(f)(x)-T(g)(x)|$$

$$= \sup_{x\in[0,1]} |T(f(x))-T(g(x))|$$

$$= \sup_{x\in[0,1]} \left|\int_0^1 (f(y)-g(y))K(x,y)\,dy\right|$$

$$\leq M \int_0^1 |f(y)-g(y)|\,dy$$

$$\leq M\|f-g\|$$

where $M = \sup_{(x,y)\in\sqcup} |K(x,y)| = |K(x_0,y_0)|$, $(x_0,y_0)\in\sqcup$ ∵ K is conti. on compact set.

∵ $|K(x,y)| < 1 \quad \forall (x,y)\in\sqcup$

∴ $M < 1$

$\Rightarrow$ T is a contraction mapping

$\Rightarrow \exists!\, f_0 \in C([0,1],\mathbb{R})$

s.t. $T(f_0) = f_0$

i.e. $f_0(x) = A(x) + \int_0^1 K(x,y) f_0(y)\,dy$ #

\# 35. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bijective.

Show that f^{-1} is continuous.

Pf: Without loss of generality, we may assume that f is strictly increasing on $\mathbb{R}$.

Let $\{y_n\} \nearrow y_0$, let $x_n = f^{-1}(y_n)$

Then x_n is also $\nearrow$;

Let $x_0 = f^{-1}(y_0)$

Claim: $x_n \nearrow x_0$ (i.e. $f^{-1}(y_n) \nearrow f^{-1}(y_0)$)

Pf: Suppose $x_n \nearrow a$.

$\because$ f is strictly increasing

$$\therefore \underset{\substack{\| \\ f(f^{-1}(y_n)) \\ \| \\ y_n}}{f(x_n)} \to f(a)$$

Since $y_n \nearrow y_0$ and $y_n \nearrow f(a)$

$\Rightarrow$ $y_0 = f(a)$

$\Rightarrow$ $\underset{\substack{\| \\ x_0}}{f^{-1}(y_0)} = a$

$\Rightarrow$ $a = x_0$.

That is $x_n \nearrow x_0$.

This implies that f^{-1} is continuous

($\because$ $y_n \nearrow y_0 \Rightarrow f^{-1}(y_n) \nearrow f^{-1}(y_0)$) #

#39.

$\because$ $f:[0,1] \to \mathbb{R}$ is continuous

$\therefore$ f is uniformly continuous on $[0,1]$.

Given $\varepsilon > 0$, choose $\delta > 0$ s.t. $|f(x)-f(y)| < \varepsilon$ as $|x-y| < \delta$.

Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{\delta}{2}$

Define

$$g(x) = \begin{cases} f(\frac{i}{n}) & \text{if } \frac{i}{n} \le x < \frac{i+1}{n}, \quad i = 0, \cdots, n-2 \\ f(\frac{n-1}{n}) & \text{if } \frac{n-1}{n} \le x \le 1 \end{cases}$$

For $x \in [0,1) \Rightarrow x \in [\frac{i}{n}, \frac{i+1}{n})$, $i = 0, \cdots, n-1$

$$\Rightarrow |g(x) - f(x)| = |f(\tfrac{i}{n}) - f(x)| < \varepsilon.$$

and $x = 1 \Rightarrow |g(1) - f(1)| = |f(\frac{n-1}{n}) - f(1)| < \varepsilon$.

$$\Rightarrow \sup_{x \in [0,1]} |g(x) - f(x)| < \varepsilon.$$

i.e. $\|g - f\| < \varepsilon$ #

#40.

a. $\delta: C([0,1], \mathbb{R}) \longrightarrow \mathbb{R}$

$f \longmapsto f(0)$

Check: δ is linear

Pf: $\alpha \in \mathbb{R}$, $f, g \in C([0,1], \mathbb{R})$

$$\begin{aligned} \delta(f+\alpha g) &= (f+\alpha g)(0) \\ &= f(0) + \alpha g(0) \\ &= \delta(f) + \alpha\,\delta(g) \ \# \end{aligned}$$

Check: δ is continuous.

Pf: Suppose $f_n \longrightarrow f$ in $C([0,1], \mathbb{R})$

Then $f_n(0) \longrightarrow f(0)$

Hence

$$\delta(f_n) \longrightarrow \delta(f).$$

Thus δ is continuous.

b. (i) prove F is continuous.

Pf. Let $f_n \longrightarrow f$ in $C([0,1], \mathbb{R})$.

We want to prove that $F(f_n) \to F(f)$ in $C([0,1], \mathbb{R})$.

Let $x \in [0,1]$, then $f_n(x) \to f(x)$.

Hence $g(f_n(x)) \longrightarrow g(f(x))$ ($\because$ g is continuous)

$\Rightarrow$ $F(f_n) \to F(f)$ as $f_n \to f$.

Thus F is continuous in $C([0,1], \mathbb{R})$. #

40.

(b) (ii) Prove that if g is uniformly continuous, then F is uniformly continuous.

Pf: Since g is uniformly continuous,

for any $\varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$|g(x) - g(y)| < \varepsilon$ as $|x-y| < \delta$.

Thus, if $\|f-h\| < \delta$, $f, h \in C([0,1], \mathbb{R})$.

then $\|F(f) - F(h)\| = \| g\circ f - g\circ h\|$

$$= \sup_{x\in[0,1]} | g(f(x)) - g(h(x))|$$

$$< \varepsilon \quad (\because \|f-h\| < \delta \Rightarrow |f(x) - h(x)| < \delta)$$

Hence F is uniformly continuous.