Exercises for § 5.3

\# 3. (i) $f_n(x) = \sqrt{n}\, x^n (1-x)$ on $[0,1]$.

$f_n'(x) = 0 \Rightarrow x = \frac{n}{n+1}$ is the critical point and also is the maximum of $f_n(x)$ on $[0,1]$.

$$\Rightarrow \quad |f_n(x)| \le \left|f_n\left(\frac{n}{n+1}\right)\right| = \sqrt{n}\left(\frac{n}{n+1}\right)^n\left(\frac{1}{n+1}\right) \le \frac{\sqrt{n}}{n+1} \quad \forall x \in [0,1]$$

$\because \frac{\sqrt{n}}{n+1} \to 0$ as $n \to \infty$,

$\therefore f_n(x) \to 0$ uniformly on $[0,1]$.

$\Rightarrow \int_0^1 f_n(x)dx \to \int_0^1 0\, dx$

(ii) $f_n'(x) = \sqrt{n}\, x^{n-1}[n-(n+1)x]$ on $(0,1)$; $f_n'(x) \to 0$ pointwise on $(0,1)$.

$f_n''(x) = 0 \Rightarrow x = \frac{n-1}{n+1} \Rightarrow f_n'\left(\frac{n-1}{n+1}\right) = \sqrt{n}\left(\frac{n-1}{n+1}\right)^{n-1} \ge \left(\frac{n-1}{n+1}\right)^{n-1}$

$$\because \lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n-1} = \frac{1}{e^2}$$

$$\therefore \lim_{n\to\infty} f_n'\left(\frac{n-1}{n+1}\right) \ge \frac{1}{e^2}$$

$\Rightarrow$ $f_n'(x) \to 0$ pointwise on $[0,1]$, but not uniformly. ※

§ 5.3

#6. (a) $\because$ $\left|\frac{x^{n-1}}{n}\right| \le |x|^{n-1} \le |1-\varepsilon|^{n-1}$ $\forall\ x \in [-1+\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$.

and $\sum_{n=1}^{\infty} |1-\varepsilon|^{n-1} < \infty$

$\therefore$ By M-test, $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ converges uniformly on $[-1+\varepsilon, 1-\varepsilon]$ for any $\varepsilon > 0$.

$\Rightarrow$ $\left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right)' = \sum_{n=1}^{\infty} \left(\frac{x^n}{n^2}\right)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ for $|x|<1$.

for $x \neq 0$, $|x|<1$

$\because$ $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$ $\forall\ x \in (-1,1)$.

$\Rightarrow$ $x \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\log(1-x)$

$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\frac{1}{x} \log(1-x)$

Thus, $\int_0^x \left(\sum_{n=1}^{\infty} \frac{t^n}{n^2}\right)' dt = \int_0^x \sum_{n=1}^{\infty} \left(\frac{t^n}{n^2}\right)' dt = -\int_0^x \frac{1}{t} \log(1-t)\,dt$.

$\because$ $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ converges uniformly on $(-1,1)$.

$$\| \quad \sum_{n=1}^{\infty} \int_0^x \left(\frac{t^n}{n^2}\right)' dt \quad \| \quad \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Hence $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{1}{t} \log(1-t)\,dt$, $|x|<1$.

6. (b)

It is actually valid at $x=-1$. To see this, recall that

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} = \frac{1}{1-x} - \frac{x^{N+1}}{1-x}.$$

Thus,

$$\sum_{n=0}^{N}\int_0^x t^n dt = \sum_{n=0}^{N}\frac{x^{n+1}}{n+1} = -\log(1-x) - \int_0^x \frac{t^{N+1}}{1-t}dt.$$

$$\parallel$$

$$x\sum_{n=1}^{N}\frac{x^{n-1}}{n}$$

Hence,

for $x\neq 0$,

$$\sum_{n=1}^{N}\frac{x^{n-1}}{n} = -\frac{1}{x}\log(1-x) - \frac{1}{x}\int_0^x \frac{t^{N+1}}{1-t}dt.$$

$$\Rightarrow \int_0^x \sum_{n=1}^{N}\frac{y^{n-1}}{n}dy = -\int_0^x \frac{1}{y}\log(1-y)dy - \int_0^x \frac{1}{y}\left(\int_0^y \frac{t^{N+1}}{1-t}dt\right)dy.$$

$$\Rightarrow \left|\sum_{n=1}^{N}\frac{(-1)^n}{n^2} + \int_0^{-1}\frac{1}{y}\log(1-y)dy\right| = \left|\int_0^{-1}\frac{1}{y}\left(\int_0^y \frac{t^{N+1}}{1-t}dt\right)dy\right|$$

$$\leq \int_0^{-1}\frac{1}{|y|}\left(\int_0^y \frac{|t|^{N+1}}{|1-t|}dt\right)dy$$

$\because -1\leq t<0$

$$\leq \int_0^{-1}\frac{1}{|y|}\underbrace{\int_0^y |t|^{N+1}dt}_{=\frac{|y|^{N+2}}{N+2}}\,dy$$

$$= \int_0^{-1}\frac{|y|^{N+1}}{N+2}dy.$$

$$\leq \frac{1}{(N+2)^2} \longrightarrow 0 \text{ as } N\to\infty$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\int_0^{-1}\frac{1}{y}\log(1-y)dy \quad ※$$

Exercise § 5.5.

\# 1. $B = \{ f \in C_b(\mathbb{R}, \mathbb{R}) \mid f(x) > 0 \text{ for all } x \in \mathbb{R} \}$.

Let $\tilde{f}(x) = \frac{1}{1+x^2} \Rightarrow \tilde{f} \in B$.

Suppose B is open in $C_b(\mathbb{R}, \mathbb{R})$.

$\Rightarrow \exists\, \varepsilon_0 > 0$ s.t. $D(\tilde{f}; \varepsilon_0) \subset B$

Define $g(x) = \tilde{f}(x) - \frac{\varepsilon_0}{2} \Rightarrow g \in C_b(\mathbb{R}, \mathbb{R})$.

and $\|\tilde{f} - g\| < \varepsilon_0 \Rightarrow g \in D(\tilde{f}; \varepsilon_0)$.

But $g \notin B$ $\rightarrow\leftarrow$.

$$\left(\begin{array}{l} \because \lim_{|x| \to \infty} \tilde{f}(x) = 0 \quad \therefore \exists\, M \in \mathbb{R}^+ \text{ s.t.} \\ \tilde{f}(|x|) < \frac{\varepsilon_0}{2} \quad \text{as } |x| > M. \\ \Rightarrow g(|x|) = \tilde{f}(|x|) - \frac{\varepsilon_0}{2} < 0. \end{array} \right)$$

$\therefore$ B is not open.

$\operatorname{Int}(B) = \{ f \in C_b(\mathbb{R}, \mathbb{R}) \mid \exists\, \delta > 0 \text{ with } f(x) > \delta \text{ for all } x \}$.

\#

5.

Let $B = \{ f_k \mid k=1,2,\dots \}$ and f_k be a convergent sequence in $C_b(A, \mathbb{R}^m)$.

Since $f_k \longrightarrow f$ in $C_b(A, \mathbb{R}^m)$.

Assume $\|f\| \le M$.

Let $\varepsilon = 1$, $\exists\ N \in \mathbb{N}$ s.t.

$$\|f_k - f\| \le 1 \quad \text{as } k \ge N.$$

$$\Rightarrow \quad \|f_k\| \le 1 + \|f\| \le 1+M.$$

Since $\|f_k\| \le M_k$ for $1 \le k \le N-1$.

Let $M' = \max\{ M_k, 1+M \}$, $1 \le k \le N-1$.

Then $\|f_k\| \le M' \quad \forall\ k$.

$\Rightarrow$ B is bounded in $C_b(A, \mathbb{R}^m)$.

If $f \in B$, then B is closed. #

Exercise § 5.6.

3.

(a) Define $I: C([0,1], \mathbb{R}) \longrightarrow \mathbb{R}$

by $I(f) = \int_0^1 f(x)\,dx$.

$\because$ I is conti on $C([0,1],\mathbb{R})$

and $(0,3)$ is open in $\mathbb{R}$.

$\therefore$ $I^{-1}(f) = \{ f \in C([0,1],\mathbb{R} \mid \int_0^1 f(x)dx \in (0,3) \}$

is open. #

(b). Let f be the accumulation point of $C_b(A, N)$.

$\Rightarrow \exists \{f_k\} \subset C_b(A,N)$ s.t. $\|f_k - f\| \to 0$ as $k \to \infty$.

i.e. $f_k \to f$ uniformly on A

$\because$ $f_k \in C_b(A,N)$, i.e. f_k is conti $\forall k$.

and $f_k \to f$ uniformly on A

$\therefore$ f is conti on A.

$\because$ $\|f_k - f\| \to 0$ as $k \to \infty$.

$\therefore$ Let $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t. $k \geq N$.

$\|f - f_k\| < 1 \Rightarrow \|f\| \leq \|f_N\| + 1 \leq M + 1$ as $\|f_N\| \leq M$.

Thus f is bounded and conti on A

$\Rightarrow$ $f \in C_b(A, N)$.

$\Rightarrow$ $C_b(A,N)$ is closed #

5.

$F_n(x) = \int_a^x f_n(t)\,dt, \quad a \le x \le b.$

$\because$ $f_n : [a,b] \to \mathbb{R}$ are uniformly bounded conti

i.e. $\exists\, M$ s.t. $\|f_n\| \le M \ \forall\, n.$

$\therefore$ $|F_n(x)| \le \int_a^x |f_n(t)|\,dt \le \int_a^x \|f_n\|\,dt \le M(b-a) \quad \forall n.$

Hence F_n are uniformly bdd on $[a,b]$. — ①

$\forall\ x, y \in [a,b], \quad x<y.$

$$|F_n(x) - F_n(y)| = \left| \int_a^x f_n(t)\,dt - \int_a^y f_n(t)\,dt \right|$$
$$= \left| \int_x^y f_n(t)\,dt \right|$$
$$\le \int_x^y \|f_n\|\,dt \le M(y-x) \le M(b-a)$$

$\Rightarrow$ given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M(b-a)}$

then $|F_n(x) - F_n(y)| < \varepsilon$ as $|x-y| < \delta. \quad \forall n.$

$\Rightarrow \{F_n\}$ is equicontinuous. — ②

By ①, ②, we get F_n has a uniformly convergent subsequence.

Exercise for Chapter 5.

22.

Let $\mathcal{B} \subset C(A, \mathbb{R}^m)$ and $A \subset \mathbb{R}^n$ be compact.

$\forall \varepsilon > 0, \forall x_0 \in A \ \exists \delta_{x_0} > 0$

s.t. $d(x, x_0) < \delta_{x_0} \Rightarrow d(f(x), f(x_0)) < \varepsilon$.

Since A is compact, $\bigcup_{x_0 \in A} D(x_0, \frac{\delta_{x_0}}{2}) \supset A$ is an open cover of A.

$\exists$ finite subcover $\bigcup_{i=1}^{N} D(x_i, \frac{\delta_i}{2}) \supset A$.

Let $\delta = \min\{\frac{\delta_i}{2}\}$.

If $x, y \in A$ and $d(x,y) < \delta$, then there exist $x_i \in A$

s.t. $d(x, x_i) < \frac{\delta_i}{2}$.

and $d(x_i, y) \leq d(x_i, x) + d(x, y)$

$< \frac{\delta_i}{2} + \frac{\delta_i}{2} = \delta_i$.

Thus $d(f(x), f(y)) \leq d(f(x), f(x_i)) + d(f(x_i), f(y))$

$< \varepsilon + \varepsilon = 2\varepsilon \qquad \forall f \in \mathcal{B}$

$\Rightarrow \mathcal{B}$ is equicontinuous. ※

30.

Let $B = \{ f \in C([0,1], \mathbb{R}) \mid f \text{ is } C^1 \text{ on } (0,1),\ f(0)=0, \text{ and } |f'(x)| \le 1 \}$.

① $\forall f \in B$

$$|f(x)| = |f(x) - f(0)| = |f'(\xi)(x-0)|$$

$$\le 1 \qquad \forall x \in [0,1]$$

$\Rightarrow$ B is bounded. $\Rightarrow$ $Cl(B)$ is bdd.

$\Rightarrow$ $Cl(B)$ is closed and bounded.

② $\forall x, y \in [0,1]$

$$|f(x) - f(y)| = |f'(\xi)||x-y|$$

$$\le |x-y|$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$

then $|f(x) - f(y)| < \varepsilon$ as $|x-y| < \delta$ $\forall f \in B$

$\Rightarrow$ B is equicontinuous.

$\Rightarrow$ $Cl(B)$ is also equicontinuous.

By ①, ② $Cl(B)$ is compact.