Exercises for §8.1

1. Prove that if R is a Riemann sum for a function f and partition P, then $L(f,P) \le R \le U(f,P)$

pf: Let P be a partition with subrectangles R_j

If $x_j \in R_j$

$$L(f,p) = \sum_j \inf(f(R_j))\, v(R_j) \le \sum_j f(x_j)\, v(R_j) = R \le \sum_j \sup(f(R_j))\, v(R_j) = U(f,p)$$

6. Let $f:[a,b] \to \mathbb{R}$ be continuous, Use Riemann's condition and uniformly continuity of f to prove that f is integrable

pf $\because$ f is continuous and $[a,b]$ is compact

$\Rightarrow$ f is bounded and f is uniformly continuous

$\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0 \ni |x_i - x_{i+1}| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$ $x, y \in [x_i\ x_{i+1}]$

$$\Rightarrow \quad U(f,P) - L(f,P) = \sum_{i=0}^{n} \left(\sup f([x_i, x_{i+1}]) - \inf f([x_i, x_{i+1}])\right) \cdot (x_{i+1} - x_i)$$

$$\le \frac{\varepsilon}{b-a} \cdot \sum_{i=1}^{n} (x_{i+1} - x_i) = \varepsilon$$

By 8.1.3 f is integrable

Exercises for § 8.2

1 Show that $\{(x, y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ has volume zero.

Pf: Let $f(t)=e^{i\pi t}$ $t\in[-1, 1]$

$\Rightarrow \{(x, y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ is the graph of f Let $x=\cos(\pi t)$ $y=\sin(\pi t)$

$\because [-1, 1]$ is compact and f is continuous $\Rightarrow$ f is uniformly continuous

$\Rightarrow \forall \varepsilon>0 \ \exists \delta \ni |x-y|<\delta \ |f(x)-f(y)|<\frac{\varepsilon}{2}$

$\Rightarrow$ partition $[-1, 1]$ into short enough intervals

Let $[-1, 1]=\bigcup_{i=1}^{m} I_i$ $\quad V(I_i)<\delta$

$\Rightarrow \{(x, y)\in\mathbb{R}^2 \mid x^2+y^2=1\} \subset \bigcup_{i=1}^{m} I_i\times[\sup f(I_i), \inf f(I_i)]$

$V(\bigcup_{i=1}^{m} I_i\times[\sup f(I_i), \inf f(I_i)]) \le \frac{\varepsilon}{2}2=\varepsilon$

$\Rightarrow \{(x, y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ has volume zero

4 Use Exercise 3 to show that the irrationals in $[0, 1]$ do not have measure

Pf: Let $A=[0, 1]\cap\mathbb{Q}$ $\quad\Rightarrow [0, 1]=A\cup B$

$B=[0, 1]\cap\mathbb{Q}^c$

Check A measure zero

$\because$ A is countable

Let $A=\{x_1\ x_2\ x_3 \cdots\}$

$\because x_i\in[x_i-\frac{\varepsilon}{2\cdot 2^i}, x_i+\frac{\varepsilon}{2\cdot 2^i}]$ $\quad\Rightarrow A\subset\bigcup_{i=1}^{\infty}(x_i-\frac{\varepsilon}{2\cdot 2^i}, x_i+\frac{\varepsilon}{2\cdot 2^i})$

$\Rightarrow \sum V(x_i-\frac{\varepsilon}{2\cdot 2^i}, x_i+\frac{\varepsilon}{2\cdot 2^i}) < \sum_{i=1}^{\infty}(\varepsilon/2^i)=\varepsilon$

$\Rightarrow A$ measure zero

If B is measure zero

By 8.2.4 $[0,1] = A \cup B$ measure zero zt

$\therefore$ B do not have measure zero

5 Must the boundary of a set have measure zero ?

Ans: No Let $A = \mathbb{Q} \cap [0,1]$

$bd(A) = [0,1]$ do not have measure zero

6 Must the boundary of a set of measure zero have measure zero ?

Ans: No Let $A = \mathbb{Q} \cap [0,1]$

A is measure zero

$bd(A) = [0,1]$ do not have measure zero

Exercises for § 8.3

4. Let $A \subseteq \mathbb{R}^n$ be open and have volume, and let $f: A \to \mathbb{R}$ be continuous, $f(x) \geq 0$, and $f(x_0) > 0$ for some $x_0 \in A$. Show that $\int_A f > 0$

pf: We may assume f is integrable

Let $f(x_0) = M$

$\because$ A is open and f is continuous

$\exists\, r > 0 \ \ni \overline{B(x_0, r)} \subset A$ and $f(x) > M - \varepsilon > 0 \quad \forall x \in \overline{B(x_0, r)}$

Let $g(x) = \begin{cases} f(x) & x \in \overline{B(x_0, r)} \\ 0 & x \in A - \overline{B(x_0, r)} \end{cases} \quad \Rightarrow g \leq f$

$\Rightarrow g$ is integrable ($\because$ $\{x \in \mathbb{R}^n \mid g(x)$ is discontinuous$\}$ measure zero

$\{x \in \mathbb{R}^n \mid g(x) \text{ is discontinuous}\} \subseteq \{(x_1, \cdots x_n) \in \mathbb{R}^n \mid \sqrt{x_1^2 + \cdots + x_n^2} = 1\}$)

$0 < \int_A g < \int_A f$

($\because 0 < (M-\varepsilon)\, v(\overline{B(x_0, r)}) < L(g, P) \leq \int_A g \leq U(g, P) \leq U(f, P)$ for any partition P

$\Rightarrow 0 < (M-\varepsilon)\, v(\overline{B(x_0, r)}) \leq \int_A g \leq \int_A f$

$\int_A g = \inf_P U(g, P) \leq \inf_P U(f, P) = \int_A g$)

6. Let $f(x) = \cos(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on $[-1, 1]$

pf: $|f(x)| = |\cos \frac{1}{x}| \leq 1$ and $[-1, 1]$ is bounded

$\lim_{x \to 0} \cos(\frac{1}{x})$ do not exist and f is continuous on $[-1, 1] \setminus \{0\}$

$\{0\}$ is measure zero

By 8.3.1 $f(x)$ is integrable on $[-1, 1]$

Exercise for § 8-4

1 If $A_1, A_2 \cdots$ have volume and $A = A_1 \cup A_2 \cdots$ is bounded. need A have volume

Ans: No Let $A = [0, 1]$

$B = A \cap \mathbb{Q} = \{x_1, x_2 \cdots x_n \cdots\}$

let $A_1 = \{x_1\}$ $A_2 = \{x_2\}$ $\cdots$

$v(\{x_1\}) = \int_A 1_{A_1} = 0$

$v\{x_2\} = \int_A 1_{A_2} = 0$

$\vdots$

But $v(B) = \int_A 1_B$ do not exist

3 Let A, B have volume and $A \cap B$ have zero volume, Use §4.1 to show that $v(A \cup B) = v(A) + v(B)$

Pf A, B, and $A \cap B$ have volume $\Rightarrow$ 1_A, 1_B and $1_{A \cap B}$ are integrable

$$1_{A \cup B} = 1_A + 1_B - 1_{A \cap B} \quad \left(\because \text{If } x \in A \cap B \quad 1_A(x) \text{ and } 1_B(x) \quad 1_{A \cap B}(x) = 1 \right)$$

$$1 = 1_{A \cup B}(x) = 1 + 1 - 1$$

By §4.1 $1_{A \cup B}$ is integrable

and $\int_{A \cup B} 1_{A \cup B} = \int_{A \cup B} 1_A + \int_{A \cup B} 1_B - \int_{A \cup B} 1_{A \cap B}$

$$\Rightarrow v(A \cup B) = v(A) + v(B) - \underset{0}{\underset{\parallel}{v(A \cap B)}} = v(A) + v(B)$$

Exercises for §8.5

2 Let $f:[a,\infty) \to \mathbb{R}$ be Riemann integrable on bounded intervals. Show that $\int_a^\infty f$ (conditional convergence) exists iff for every $\varepsilon > 0$ there is a T such that $t_1, t_2 > T$ implies

$$\left| \int_{t_1}^{t_2} f(x)\,dx \right| < \varepsilon$$

(this is called the Cauchy criterion)

Pf: "$\Rightarrow$" $\int_a^\infty f$ exists $\Rightarrow$ $\lim_{b\to\infty} \int_a^b f$ exist

$$\Rightarrow \exists N \in \mathbb{N} \ni b > N \Rightarrow \left| \int_a^b f - \int_a^\infty f \right| < \frac{\varepsilon}{2}$$

Let $T = N$ $\Rightarrow$ $\left| \int_{t_1}^{t_2} f(x)\,dx \right| = \left| \int_a^{t_2} f(x)\,dx - \int_a^{t_1} f(x)\,dx \right|$

$$\leq \left| \int_a^{t_2} f(x)\,dx - \int_a^\infty f(x) \right| + \left| \int_a^\infty f(x) - \int_a^{t_1} f(x)\,dx \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad t_1, t_2 > T$$

"$\Leftarrow$" Let $F_n = \int_a^n f(x)\,dx$

for $\varepsilon > 0$ $\exists T$ s.t. $n, m > T \Rightarrow \left| \underset{F_n}{\underset{\shortparallel}{\int_a^n f(x)\,dx}} - \underset{F_m}{\int_a^m f(x)\,dx} \right| < \varepsilon$

$\Rightarrow F_n$ is a Cauchy sequenec

$\exists M \ni F_n \to M$ as $n \to \infty$

Check $\lim_{b\to\infty} \int_a^b f = M$

$\because F_n \to M$

$\forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N} \ni \left| \int_a^n f - M \right| < \frac{\varepsilon}{2}$

and $\exists T \ni \left| \int_{t_1}^{t_2} f \right| < \frac{\varepsilon}{2}$

Let $N = \max\{N_1, T\}$

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N} \ni \left| \int_a^b f - M \right| \leq \left| \int_a^b f - \int_a^n f \right| + \left| \int_a^n f - M \right|$ $\quad n, b > N$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \lim_{b\to\infty} \int_a^b f = M$

$\Rightarrow \int_a^\infty f$ exist

5 For what α is $\int_0^\infty \frac{x^\alpha}{1+x^\alpha}dx$ convergent ?

Ans: $\int_0^\infty \frac{x^\alpha}{1+x^\alpha}dx = \int_0^1 \frac{x^\alpha}{1+x^\alpha}dx + \int_1^\infty \frac{x^\alpha}{1+x^\alpha}dx$

$\because \int_0^1 \frac{x^\alpha}{1+x^\alpha}dx < \int_0^1 x^\alpha dx$ exist

$\Rightarrow \int_0^1 \frac{x^\alpha}{1+x^\alpha}dx$ exist

$\int_1^\infty \frac{x^\alpha}{1+x^\alpha}dx < \int_1^\infty x^\alpha dx$

By 8.5-7 $\alpha < -1$ $\int_1^\infty \frac{x^\alpha}{1+x^\alpha}dx$ exist

For $\alpha \geq -1$

Check $\frac{x^\alpha}{1+x^\alpha} > \frac{1}{x}$ for x large enough

$\alpha > 0$ $\lim_{x\to\infty} \frac{x^\alpha}{1+x^\alpha} = 1 \Rightarrow \exists N_1 \in \mathbb{N}$ $\frac{x^\alpha}{1+x^\alpha} > \frac{1}{x}$ $x \geq N_1$

$\alpha = 0$ $\lim_{x\to\infty} \frac{1}{1+1} = \frac{1}{2} \Rightarrow \exists N_2 \in \mathbb{N}$ $\frac{x^\alpha}{1+x^\alpha} > \frac{1}{x}$ $x \geq N_2$

$\alpha < 0$
If $\frac{x^\alpha}{1+x^\alpha} > \frac{1}{x} \Rightarrow$ $x^{\alpha+1} > 1 + x^\alpha$

$x^{\alpha+1} - x^\alpha > 1$

$\because \alpha + 1 > 0$

$\lim_{x\to\infty} x^{\alpha+1} - x^\alpha = \infty$

$\Rightarrow \exists N_3 \in \mathbb{N}$ $\frac{x^\alpha}{1+x^\alpha} > \frac{1}{x}$ $x \geq N$

Let $N = \max\{N_1, N_2, N_3\}$

$\Rightarrow$ If $\alpha \geq -1$ $= \int_1^\infty \frac{x^\alpha}{1+x^\alpha} = \int_1^N \frac{x^\alpha}{1+x^\alpha}dx + \int_N^\infty \frac{x^\alpha}{1+x^\alpha}dx \geq \int_1^N \frac{x^\alpha}{1+x^\alpha}dx + \underbrace{\int_N^\alpha \frac{1}{x}dx}_{=\infty}$

$\therefore \alpha \geq -1$ $\int_0^\infty \frac{x^\alpha}{1+x^\alpha}dx$ do not exist

Exercises for § 8-6

2 Evaluate $\lim_{n\to\infty} \int_0^1 \frac{e^x \sin nx}{n} dx$

Ans

$$\left|\frac{e^x \sin nx}{n}\right| \le \left|\frac{e^x}{n}\right| \quad (\because |\sin nx| \le 1)$$

$$\Rightarrow \quad \because \left|\frac{e^x}{n}\right| = \frac{e^x}{n} \ge 0 \cdot x \in [0,1]$$

$$\frac{e^x}{n} \le \frac{e^x}{n+1} \quad \text{and} \quad \lim_{n\to\infty} \frac{e^x}{n} = 0$$

By 8.6.1 $\lim_{n\to\infty} \int_0^1 \left|\frac{e^x}{n}\right| dx = \lim_{n\to\infty} \int \frac{e^x}{n} dx = 0$

$$\Rightarrow \because \left|\frac{e^x \sin nx}{n}\right| \le \left|\frac{e^x}{n}\right|$$

$$\Rightarrow \left|\int \frac{e^x \sin nx}{n} dx\right| \le \int \left|\frac{e^x \sin nx}{n}\right| dx \le \int \left|\frac{e^x}{n}\right| dx$$

$$\Rightarrow 0 \le \lim_{n\to\infty} \left|\int \frac{e^x \sin nx}{n} dx\right| \le \lim_{n\to\infty} \left|\frac{e^x \sin nx}{n}\right| dx \le \lim_{n\to\infty} \int \left|\frac{e^x}{n}\right| dx = 0$$

$$\Rightarrow \left|\int \frac{e^x \sin x}{n} dx\right| = 0 \quad \text{as} \quad n \to 0$$

$$\Rightarrow \lim_{n\to\infty} \int \frac{e^x \sin nx}{n} dx = 0$$

3 Evaluate $\lim_{n\to\infty} \int_0^1 \frac{1-e^{-nx}}{\sqrt{x}} dx$

Ans Let $f_n(x) = \frac{1-e^{-nx}}{\sqrt{x}}$ $\quad \because \int_0^1 \frac{1}{\sqrt{x}} = 2$

$0 \le f_n(x) \le f_{n+1}(x) \le \frac{1}{\sqrt{x}}$ and $f_n(x) \to \frac{1}{\sqrt{x}}$ $\quad$ By 8.6.2 $\lim_{n\to\infty} \int_0^1 \frac{1-e^{-nx}}{\sqrt{x}} dx = 2$

Exercises for §8-1

2 Let T_n and T be distribution. Say $T_n \to T$ if $T_n(f) - T(f)$ for all $f \in D$

Show that

$$\sqrt{\frac{n}{\pi}}\, e^{-nx^2} \to \delta$$

Pf: for all $f \in D$ check: $\lim_{n\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} dx = \int_{-\infty}^{\infty} \delta(x) f(x) = f(0)$

$\sqrt{n}x = y$

$$\int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy = 1$$

$$\int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} (f(x) - f(0)) dx$$

$\because$ f is continuous at 0

$\forall \varepsilon > 0\ \exists \delta > 0 \ni |f(x) - f(0)| < \varepsilon$ as $|x| < \delta$

$$= \int_{-\delta}^{\delta} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} (f(x) - f(0)) dx + \int_{(-\infty,\infty)\setminus(-\delta,\delta)} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} (f(x) - f(0)) dx$$

$$< \varepsilon \underbrace{\int_{-\delta}^{\delta} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} dx}_{\leq 1} + 2M \int_{(-\infty,\infty)\setminus(-\sqrt{n}\delta,\sqrt{n}\delta)} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$$

$\because$ $f \in C^\infty$ and identically zero outside interval

$$< \varepsilon + \frac{2M}{\sqrt{\pi}} \int_{(-\infty,\infty)\setminus(-\sqrt{n}\delta,\sqrt{n}\delta)} e^{-y^2} dy$$

$\Rightarrow f$ is bounded

$\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ and $e^{-y^2} > 0$ $\lim_{a\to\infty} \int_{-a}^{a} e^{-y^2} dy = \sqrt{\pi}$

$$\Rightarrow \int_{(-\infty,\infty)\setminus(-\sqrt{n}\delta,\sqrt{n}\delta)} e^{-y^2} dy \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} (f(x) - f(0)) \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \to \delta$$

3 If $T_n \to T$ (see Exercise 2). Show that $T_n' \to T'$. Discuss and compare with §5-8

Pf: (1) $T_n'(f) = -T_n(f') \to -T(f') = T'(f)$ for each $f \in D$

$\Rightarrow T_n' \to T'$

(2) $T_n \to T \Rightarrow T_n' \to T'$

§5-3 $f_n \to f$ and $f_n' \to g$ uniformly $\Rightarrow f_n' \to f' = g$

4. Find a sequence of continuous functions g_n such that $g_n \to \delta'$

Ans By exercise 3 $T_n \to T \Rightarrow T_n' \to T'$

By exercise 2 $\underbrace{\sqrt{\frac{n}{\pi}} e^{-nx^2}}_{= T_n} \to \delta$

$$T_n' \to \delta'$$
$$T_n' = \underline{\sqrt{\frac{n}{\pi}} e^{-nx^2} \cdot (-2nx)}$$

Exercises for Chapter 8

1\. a. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$, where A is bounded and f is bounded and integrable over A. Consider another bounded integrable function $g: A \to \mathbb{R}$ such that $g(x) = f(x)$ except on a set $S \subset A$ of measure zero. Assume that f and g are integrable on S and $A \setminus S$. Prove that $\int_A g = \int_A f$

b. If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ and $g: A \subset \mathbb{R}^n \to \mathbb{R}$ are bounded functions, integrable on bounded set A, and $\int_A |f-g| = 0$. Prove that $f(x) = g(x)$ for all $x \in A$, except possibly for a set of measure zero

pf: a. By 8.4.1

$$\int_A f = \int_{A\setminus S} f + \underbrace{\int_S f}_{\substack{=0 \\ \text{By } 8.3.4}} = \int_{A\setminus S} f = \int_{A\setminus S} g = \int_{A\setminus S} g + \underbrace{\int_S g}_{=0} = \int_A g$$

b. By 8.3.4

$\{x \in A \mid |f(x) - g(x)| \neq 0\}$ has measure

$=$

$\{x \in A \mid f(x) \neq g(x)\}$

3 Prove that an increasing function $f: [a, b] \to \mathbb{R}$ is Riemann integrable.

pf Partition $[a, b]$ by $x_j = a + (j/n)(b-a)$ with $j = 0, 1, 2, \cdots, n$

$$\Rightarrow U(f, P) = f(x_1)\frac{b-a}{n} + f(x_2)\frac{b-a}{n} + \cdots + f(x_n)\frac{b-a}{n} = (f(x_1) + \cdots + f(x_n))\frac{b-a}{n}$$

$$L(f, P) = (f(x_0) + \cdots + f(x_{n-1}))\frac{b-a}{n}$$

$$U(f, P) - L(f, P) = (f(b) - f(a))\frac{(b-a)}{n} \to 0 \text{ as } n \to \infty$$

By 8.1.3 f is Riemann integrable

7 Let $f:[a,b]\to\mathbb{R}$ be continuous and differentiable on (a,b). Assume that $f(a)=0$, $f(b)=-1$, and $\int_a^b f(x)=0$. Prove that there is a $c\in(a,b)$ such that $f'(c)=0$

Pf: $\because$ $f(b)=-1$ and f is continuous

$\Rightarrow \exists\delta>0 \ni f(x)<0 \quad x\in[b-\delta, b] \Rightarrow \int_{[b-\delta,b]} f<0$

$\Rightarrow \exists x_0\in(a,b) \ni f(x_0)>0$

$f(x_0)>0$ and $f(b)<0$ By intermediate value theorem

$\exists x_1$ with $x_0<x_1<b$ $f(x_1)=0$

By Mean Value Theorem $\exists c\in(a,x_1) \ni f'(c)=0$

12 Prove that A has measure zero iff for every $\varepsilon>0$ there is a covering of A by sets $V_1, V_2, \cdots$ with volume such that $\sum_{i=1}^{\infty} v(V_i)<\varepsilon$

Pf "$\Rightarrow$" By Definition A is measure zero if $\forall\varepsilon>0$, there is a covering of A say $S_1, S_2, \cdots$ by a countable (or finite) number of rectangle such that $\sum_{i=1}^{\infty} v(S_i)<\varepsilon$

Let $V_1=S_1$ $V_2=S_2$ $\cdots$

"$\Leftarrow$" Claim: If A has volume, then for every $\varepsilon>0$, there is finite covering of A by rectangles say $S_1 \cdots S_m$ such that $\sum_{i=1}^{m} v(S_i)-v(A)<\varepsilon$

Pf $v(A)=\int_A 1_A(x)dx$ Let S be a closed rectangle containing A

1_A be the characteristic function of A

$\exists P_0$ be the collection of all those subrectangles S_i that intersect A

$\Rightarrow U(1_A, P)=\sum_{S\in P_0} v(S)-\int_A 1_A(x)dx<\varepsilon$

$=\sum_{S\in P_0} v(S)-v(A)$

By the claim we can find finite rectangles S_{ij} $j=1\cdots m$

$$\bigcup_{j=1}^{m} S_{ij} > V_i \text{ and } \sum_{j=1}^{m} v(S_{ij}) - v(V_i) < \frac{\varepsilon}{2^i}$$

$\forall \varepsilon > 0$ there is a covering of A by sets V_1 V_2 with volume s.t. $\sum_{i=1}^{\infty} v(V_i) < \frac{\varepsilon}{2}$

$$\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m} S_{ij} \supseteq \bigcup_{i=1}^{\infty} V_i \supseteq A$$

$$\therefore \sum_{i=1}^{\infty} \sum_{j=1}^{m} v(S_{ij}) \le \sum_{i=1}^{\infty} v(V_i) + \frac{\varepsilon}{2^i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow$ A is measure zero

16.c Find integrable functions $f_k : A \to \mathbb{R}$ such that $f_k \to f$ pointwise but f is not integrable

Ans Let $f_k = \chi_{B(0,k)} \to \chi_{\mathbb{R}^n} = f$

$\int f_k$ is integrable

$\int f$ not exist

18 a. Generalize the mean value theorem for integrals (8.4.1.vi) to the case where A is any bounded connect set

b If $\varphi(x) \geq 0$ for $x \in$ a connected compact set $A \subset \mathbb{R}$, φ is continuous and increasing in x, and f is positive and integrable, then $f\varphi$ is integrable and $\int_A f\varphi = \varphi(c)\int_A f$ for some point $c \in A$ (second mean value theorem).

c Show that b fails if A is bounded not compact.

Pf: a let $m = \inf\{f(x) \mid x \in A\}$ and $M = \sup\{f(x) \mid x \in A\}$ fix

If $m = -\infty$ $M = +\infty$ $\exists N \in \mathbb{N} \ni -N \leq \frac{(\int_A f)}{v(A)} \leq N$ and $\exists x_0\ x_1$ $f(x_0) \leq -N$

$f(x_1) \geq N$

By intermediate value theorem $\exists x_3 \ni f(x_3) = \frac{\int_A f}{v(A)} \Rightarrow \int_A f = f(x_3)\,v(A)$

b

$\because$ A is compact $\Rightarrow \sup\{\varphi(x) \mid x \in A\} = M < \infty$
$\inf\{\varphi(x) \mid x \in A\} = m \geq 0$

$\Rightarrow \varphi f \leq Mf$ integrable $\Rightarrow \varphi f$ is integrable

$\because m\int f \leq \int \varphi f \leq \int Mf$

Let $g(x) = \varphi(x)\int_A f$ $\because \varphi$ is continuous $\Rightarrow g$ is continuous

$\because$ A is compact $\exists x_0, y_0 \ni \varphi(x_0) = M$ $\varphi(y_0) = m$

By intermediate value theorem

$\exists c \ni g(c) = \varphi(c)\int_A f = \int \varphi f$

c Let $A = (0, 1) \Rightarrow A$ is not compact

$f(x) = 1$ $x \in A$ $\varphi(x) = \frac{1}{1-x}$ $\Rightarrow \varphi f$ is not integrable $\left(\because \int_0^1 \frac{1}{1-x}\,dx = \infty\right)$

22 The gamma function is defined to be the function given by the improper integral $\Gamma(p) = \int_1^\infty e^{-x}x^{p-1}\,dx$. Show that the integral is convergent for $p>0$

pf:

$$\lim_{x\to\infty} e^{-x}x^{p+1} = \lim_{x\to\infty} \frac{x^{p+1}}{e^{x}} = 0 \quad \text{By l'Hopital's rule}$$

$\Rightarrow \exists M\in\mathbb{N} \ni |e^{-x}x^{p+1}| < 1$ when $x \geq M$

$\Rightarrow e^{-x}x^{p-1} < \frac{1}{x^2}, \quad x \geq M$

$$\int_1^M e^{-x}x^{p-1}\,dx + \int_M^\infty e^{-x}x^{p-1}\,dx$$

$\because e^{-x}x^{p-1}$ is continuous $\Rightarrow \int_1^M e^{-x}x^{p-1}\,dx$ exist

$$\int_M^\infty e^{-x}x^{p-1} \leq \int_M^\infty \frac{1}{x^2}\,dx < \infty$$

$\Rightarrow \int_1^\infty e^{-x}x^{p-1}\,dx$ exist for $p>0$

27 Prove that if $f: A\subset\mathbb{R}^n \to \mathbb{R}$ is continuous. A is open with volume and $\int_B f = 0$ for each $B\subset A$ with volum, then $f=0$

pf: If $\exists x_0\in A$ $f(x_0)\neq 0$

$\because f$ is continuous

$\Rightarrow \exists \delta>0 \ni \forall x\in D(x_0,\delta)$ $f(x)>0$

$\Rightarrow \int_{D(x_0,\delta)} f(x)\,dx > 0 \neq 0 \rightarrow\leftarrow \quad \Rightarrow f=0$

31 Let $f: [a,b] \to \mathbb{R}$ be differentiable and assume that f' is integrable. Prove that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Pf Let $a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a,b]$

By Mean Value Theorem $\exists\ t_j \in [x_j, x_{j+1}] \ni f(x_{j+1}) - f(x_j) = f'(t_j)(x_{j+1} - x_j)$

$$\Rightarrow f(b) - f(a) = \sum_{j=0}^{n-1} (f(x_{j+1}) - f(x_j)) = \sum_{j=0}^{n-1} f'(t_j)\cdot(x_{j+1} - x_j)$$

$\because f'$ is integrable $\sum_{j=1}^{n-1} f'(t_j)(x_{j+1} - x_j) \to \int_a^b f'(x)\,dx$ as $\max_j \{x_{j+1} - x_j\} \to 0$

$$\Rightarrow f(b) - f(a) = \int_a^b f'(x)\,dx$$

33 Define a function f on the interval $[0,1]$ by putting $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. Show that $|f|$ is integrable on $[0,1]$ but f is not.

Pf (1) Let $g(x) = |f(x)| = 1 \quad \forall x \in [0,1]$

$\Rightarrow g$ is continuous on $\underbrace{[0,1]}_{\text{compact}} \Rightarrow \underbrace{g}_{|f|}$ is integrable

(2) $\because \mathbb{Q} \cap [0,1]$ dense in $[0,1]$ and $\mathbb{Q}^c \cap [0,1]$ dense in $[0,1]$

$\Rightarrow$ for any $x \in [0,1]$

for any $\delta > 0$ $\exists\ y_1$ is rational y_2 is irrational $\ni y_1, y_2 \in D(x,\delta)$

$\Rightarrow f(x)$ is not continuous at x

$\Rightarrow B = \{x \mid f \text{ is not continuous at } x\} = [0,1] \quad v(B) = 1$

By Lebesgue's Theorem f is not integrable

35 Let $A_n = [(n+1)+(n+2)+\cdots+(n+n)]/n$. Prove $\lim_{n\to\infty}(\frac{1}{n})A_n = \frac{3}{2}$, using the Riemann integral

Pf

$$(\tfrac{1}{n})A_n = [(1+\tfrac{1}{n})+(1+\tfrac{2}{n})+\cdots+(1+\tfrac{n}{n})]$$

$$= \sum_{j=1}^{n}(1+\tfrac{j}{n})(\tfrac{1}{n})$$

let $f(x) = 1+x$

$$\Rightarrow \sum_{j=1}^{n}(1+\tfrac{j}{n})(\tfrac{1}{n}) = U(f,P) \to \int_0^1 (1+x)\,dx \quad \text{as } n\to\infty$$

$$= x+\tfrac{1}{2}x^2\Big|_0^1$$

$$= 1+\tfrac{1}{2} = \tfrac{3}{2}$$

36 Prove that $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} = e^{-1}$ by considering Riemann sums for $\int_0^1 \log x\,dx$ based on partition $\frac{1}{n} < \frac{2}{n} < \cdots < 1$

Pf:

$$\frac{(n!)^{\frac{1}{n}}}{n} = e^{\ln\frac{(n!)^{\frac{1}{n}}}{n}} = e^{\frac{1}{n}(\ln n! - n\ln n)}$$

$$= e^{\frac{1}{n}[(\ln 1-\ln n)+(\ln 2-\ln n)+\cdots+(\ln n-\ln n)]}$$

$$= e^{\frac{1}{n}(\ln\frac{1}{n}+\ln\frac{2}{n}+\cdots+\ln\frac{n}{n})}$$

$$\Rightarrow \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} e^{\ln\frac{(n!)^{\frac{1}{n}}}{n}} = e^{\int_0^1 \ln x\,dx} = e^{-1}$$

$$\left(\begin{aligned}\int_0^1 \ln x\,dx = \lim_{a\to 0}\int_a^1 \ln x\,dx &= \lim_{a\to 0}\left(x\ln x\Big|_a^1 - \int_a^1 1\,dx\right)\\ &= \lim_{a\to 0}\left(a\ln a-(1-a)\right)\\ &= -1\end{aligned}\right)$$

38 Let $f:[0,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}0 & \text{if } x \text{ is irrational}\\ 1/q & \text{if } x=p/q\end{cases}$$

where $p,q\geq 0$ with no common factor. Show that f is integrable, and compute $\int_0^1 f$ (See also Exercise 34)

Pf: Claim: f is continuous on $[0,1]\cap\mathbb{Q}^c$

Pf: Let $x\in[0,1]\cap\mathbb{Q}^c$

If $\{x_n\}\subset\mathbb{Q}^c\cap[0,1]$ and $x_n\to x$

$\Rightarrow \lim_{n\to 0} f(x_n)=0=f(x)$

If $\{x_n\}\subset\mathbb{Q}\cap[0,1]$ and $x_n\to x$

$\because x_n\to x \to \{x_n\}$ is Cauchy sequence

let $x_n=\frac{p_n}{q_n}$ Claim $q_n\to\infty$ as $n\to\infty$

If not $\exists M\in\mathbb{N}$ $\{q_{n_k}\}\subseteq\{q_n\}$ and $q_{n_k}\leq M$ all $k\in\mathbb{N}$

Consider $\{\frac{p_{n_k}}{q_{n_k}}\}$ $\because q_{n_k}\leq M$ and $\frac{p_{n_k}}{q_{n_k}}\leq 1$

$\therefore \{\frac{p_{n_k}}{q_{n_k}}\}$ only have finite numbers $\Rightarrow \{\frac{p_{n_k}}{q_{n_k}}\}$ is not Cauchy sequence

$\because \{x_n\}$ is Cauchy sequence $\Rightarrow \{\frac{p_{n_k}}{q_{n_k}}\}$ is Cauchy sequence $\times$

$\Rightarrow q_n\to\infty \Rightarrow \lim_{x\to 0} f(x_n)=0=f(x)$

$\Rightarrow f$ is continuous on $[0,1]\cap\mathbb{Q}^c$

$\{x\in[0,1] \mid f(x) \text{ is not continuous}\}\subseteq[0,1]\cap\mathbb{Q}$ and $[0,1]\cap\mathbb{Q}$ measure zero

By Lebesque's Theorem f is integrable.

$\because \int_{[0,1]} f$ exist $\Rightarrow \int_{[0,1]\cap\mathbb{Q}} f$ exists and $\int_{[0,1]\cap\mathbb{Q}^c} f$ exist

$\left(\int_{[0,1]} \underbrace{f 1_{[0,1]\cap\mathbb{Q}}}_{\le f \text{ By } 8.4.1} \right) \qquad \left(\int_{[0,1]} f 1_{[0,1]\cap\mathbb{Q}^c} \right)$

By 8.4.1

$$\int_{[0,1]} f = \underbrace{\int_{[0,1]\cap\mathbb{Q}} f}_{(8.3.4\ i)} + \int_{[0,1]\cap\mathbb{Q}^c} f = 0 + 0 = 0$$

$\because f(x) = 0 \quad x \in [0,1]\cap\mathbb{Q}^c$

20 Suppose $f:(0,b]\to\mathbb{R}$ is continuous, positive, and integrable on $(0,b]$ and as $x\to 0$ from the right, $f(x)$ increases monotonically to $+\infty$. Prove that $\varepsilon f(\varepsilon)\to 0$ as $\varepsilon\to 0$

Pf: $\because$ f is integrable on $(0,b]$ $\Rightarrow$ $\int_{\frac{x}{2}}^{x} f(x)\,dx \to 0$ as $x\to 0$ $x\in(0,b]$

$\because$ f increases monotonically to $+\infty$ from the right

$\Rightarrow \frac{x}{2}f(x) \leq \int_{\frac{x}{2}}^{x} f(x)$

$\Rightarrow \frac{x}{2}f(x) \to 0$ as $x\to 0$

$\Rightarrow x f(x) \to 0$ as $x\to 0$

47 For every $\alpha>0$, compare $\int_0^N x^\alpha dx$ with $\sum_{n=0}^{N} n^\alpha$ and $\sum_{n=0}^{N-1} n^\alpha$ and hence determine

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{n^\alpha}{N^{1+\alpha}}$$

Ans. Let $x_0=0$ $x_1=1$ $x_2=2$ $\cdots x_n=N$ be a partition

$f(x)=x^\alpha$

$\Rightarrow \sum_{n=1}^{N} n^\alpha = U(f,P) \geq \int_0^N x^\alpha dx \geq L(f,P)=\sum_{n=0}^{N-1} n^\alpha$

$\Rightarrow \frac{1}{N^{1+\alpha}}\sum_{n=1}^{N} n^\alpha \geq \frac{1}{N^{1+\alpha}}\int_0^N x^\alpha dx \geq \frac{1}{N^{1+\alpha}}\sum_{n=0}^{N-1} n^\alpha$

$\Rightarrow \frac{1}{N^{1+\alpha}}\sum_{n=1}^{N} n^\alpha \geq \frac{1}{1+\alpha} \geq \frac{1}{N^{1+\alpha}}\sum_{n=0}^{N-1} n^\alpha$ $\forall N\in\mathbb{N}$

$\frac{1}{N^{1+\alpha}}\sum_{n=1}^{N} n^\alpha - \frac{1}{N^{1+\alpha}}\sum_{n=0}^{N-1} n^\alpha = \frac{1}{N^{1+\alpha}}N^\alpha = \frac{1}{N} \to 0$ as $N\to 0$

$\Rightarrow \lim_{N\to\infty}\sum_{n=1}^{N}\frac{n^\alpha}{N^{1+\alpha}} = \lim_{N\to\infty}\sum_{n=0}^{N-1}\frac{n^\alpha}{N^{1+\alpha}} = \frac{1}{1+\alpha}$

48 For any function f(x) continuous over the reals, define the sequence $f_n(x) = n\int_x^{x+\frac{1}{n}} f(s)\,ds$ for $n = 1, 2, 3 \ldots$. Show that $\frac{df_n(x)}{dx}$ exists even if $\frac{df(x)}{dx}$ does not, that $f(x) = \lim_{n\to\infty} f_n(x)$, and that convergence to the limit is uniformly when f is uniformly continuous.

Pf: (1)

$$\frac{df_n(x)}{dx} = \lim_{\alpha\to 0} \frac{n\int_{x+\alpha}^{x+\alpha+\frac{1}{n}} f(s)\,ds - n\int_x^{x+\frac{1}{n}} f(s)\,ds}{\alpha}$$

If $\alpha > 0$ $\because \alpha \to 0$ $\exists\, \alpha > 0 \ni \alpha < \frac{1}{n} \Rightarrow x+\alpha < x+\frac{1}{n}$

$$n\int_{x+\alpha}^{x+\alpha+\frac{1}{n}} f(s)\,ds - n\int_x^{x+\frac{1}{n}} f(s)\,ds$$

$$= n\int_{x+\frac{1}{n}}^{x+\alpha+\frac{1}{n}} f(s)\,ds - n\int_x^{x+\alpha} f(s)\,ds$$

$$\lim_{\alpha\to 0} \frac{n\int_{x+\alpha}^{x+\alpha+\frac{1}{n}} f(s)\,ds - n\int_x^{x+\frac{1}{n}} f(s)\,ds}{\alpha} = \lim_{\alpha\to 0} \frac{n\left(\int_{x+\frac{1}{n}}^{x+\delta+\frac{1}{n}} f(s)\,ds - \int_x^{x+\delta} f(s)\,ds\right)}{\alpha}$$

By Mean Value Theorem for Integrals

$$= \lim_{\alpha\to 0} \frac{n(f(c_1)\delta - f(c_2)\delta)}{\alpha}$$

$$= \lim_{\alpha\to 0} n(f(c_1) - f(c_2)) = 0$$ ($\because$ f is continuous $f(c_1) \to f(c_2)$ as $c_1 \to c_2$)

If $\alpha < 0$

$$n\int_{x+\delta}^{x+\delta+\frac{1}{n}} f(s)\,ds - n\int_x^{x+\frac{1}{n}} f(s)\,ds$$

$$= n\left(\int_{x+\delta}^{x} f(s)\,ds - \int_{x+\delta+\frac{1}{n}}^{x+\frac{1}{n}} f(s)\,ds\right) \quad \text{as } |\alpha| < \tfrac{1}{n}$$

$$\lim_{\alpha\to 0} \frac{n\int_{x+\alpha}^{x+\alpha+\frac{1}{n}} f(s)\,ds - n\int_x^{x+\frac{1}{n}} f(s)\,ds}{\alpha} = \lim_{\alpha\to 0} \frac{n\left(\int_{x+\delta}^{x} f(s)\,ds - \int_{x+\delta+\frac{1}{n}}^{x+\frac{1}{n}} f(s)\,ds\right)}{\alpha}$$

By Mean Value Theorem for Integrals $= \lim_{\alpha\to 0} \frac{n(f(c_3)(-\delta) - f(c_4)(-\delta))}{\alpha}$

$= 0$ ($\because$ f is continuous $f(c_3) \to f(c_4)$ as $c_3 \to c_4$)

$\Rightarrow \frac{df_n(x)}{dx}$ exist

(2) $f_n(x) = n\int_x^{x+\frac{1}{n}} f(\xi)\, d\xi = n\cdot\frac{1}{n} f(\xi_0) \quad \xi_0 \in [x, x+\frac{1}{n}]$

$\to f(x)$ as $\frac{1}{n}\to 0$ ($\because f$ is continuous)

f is uniform continuous

$\Rightarrow \forall \varepsilon>0\ \exists \delta>0$ st $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon \quad \forall x, y\in\mathbb{R}$

for any $x\in\mathbb{R}$ Let $\frac{1}{n}<\delta$

$|f_n(x)-f(x)| = |f(c)-f(x)|<\varepsilon \quad c\in[x, x+\frac{1}{n}] \Rightarrow |c-x|<\delta$

$\Rightarrow f_n \to f$ uniformly

50 State whatever lemmas, theorem, and so forth are needed to justify each of the following assertions:

a $\lim_{n\to\infty} \sum_{k=1}^{\infty} 2^{-k} \sin(k/n) = 0$

b If $f(x)$ is a power series converging in $(-1,1)$, then the same is true for $f'(x)$

c Let $f(x)=\tan(\pi x/2)$ and set $a_n = f^{(n)}(0)/n!$ Then $\sum_{n=0}^{\infty} a_n$ is not a convergent series.

(Do not attempt to compute a_n)

d If $f_n(x)$ is differentiable on $[a,b]$ with $|f_n'(x)|<10$ for all n and if $x\in[a,b]$ and $f_n(x)\to 0$ at each x, then $f_n(x)\to 0$ uniformly

e $f(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{3^k}$ has a continuous derivative

f $|e^x - 1 - x - x^2/2 - \cdots - x^{99}/99!| \le e^x x^{100}/100!$ for $x>0$

g $f(x) = \sum_{n=1}^{\infty} x^n/n^3$ is continuous in the closed interval $[-1,1]$

h $\lim_{x\to 0} \frac{e^{ax}-1}{\sin bx} = \frac{a}{b}$

Ans: a Let $g_k(x) = 2^{-k} \sin\left(\frac{k}{x}\right)$, $x\in[1,\infty)$

$|g_k(x)| \le \frac{1}{2^k}$

By M test $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly $\because g_k$ is continuous

$\Rightarrow \sum_{k=1}^{\infty} g_k(x)$ is continuous $\qquad \sin\left(\frac{k}{x}\right)\to 0$ as $x\to\infty$

$\lim_{x\to\infty} \sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{\infty} \lim_{x\to\infty} g_k(x) \quad \because \lim_{x\to\infty} g_k(x) = \lim_{x\to\infty} 2^{-k}\sin\left(\frac{k}{x}\right) = 0$

$\Rightarrow \sum_{k=1}^{\infty} \lim_{x\to\infty} g_k(x) = \sum_{k=1}^{\infty} 0 = 0$

b 5-10.3

c $f(x) = \tan(\pi x/2) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f(1) = \tan\left(\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} a_n$ divergent d By Fundamental Theorem of Calculus

$$\int_x^y f_n'(t)\,dt = f_n(x) - f_n(y) \qquad \forall n \in \mathbb{N}$$

$\forall \varepsilon > 0 \quad \exists \delta = \frac{\varepsilon}{10} \ni |x-y| < \delta \quad |f_n(x) - f_n(y)| < 10 \cdot \frac{\varepsilon}{10} = \varepsilon \quad \forall n \in \mathbb{N}$

$\Rightarrow \{f_n\}$ equicontinuous

$\Rightarrow \forall \varepsilon > 0 \quad \exists \delta > 0 \ni |x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N}$

$\because [a,b]$ is compact $\quad \bigcup_{x \in [a,b]} D(x,\delta) \supset [a,b]$

$\exists\, x_1 \cdots x_n \ni \bigcup_{i=1}^{n} D(x_i,\delta) \supset [a,b]$

$f_n(x_i) \to 0$ as $n \to 0$

$\Rightarrow \forall \varepsilon > 0 \quad \exists N_i \in \mathbb{N} \ni |f_n(x_i) - 0| < \frac{\varepsilon}{3}$

Let $N = \max\{N_1 \cdots N_n\}$

$\forall \varepsilon > 0$

For any $x \in [a,b]$: $\quad x \in D(x_i,\delta)$ for some $x_i \qquad f(x) = 0$

$$|f_n(x) - 0| \le |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)|$$
$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ when } n \ge N$$

$\Rightarrow f_n \to 0$ uniformly

e

$$\Bigg|\underbrace{\frac{\cos(2^k x)}{3^k}}_{= f_n(x)}\Bigg| \le \left|\frac{1}{3^k}\right| \quad \text{By M-test } \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{3^k} \text{ convergent uniformly}$$

$$|f_n'(x)| = \left|\frac{-\sin(2^k x)\, 2^k}{3}\right| = \left|-\left(\frac{2}{3}\right)^k \sin(2^k x)\right| < \left(\frac{2}{3}\right)^k$$

By M-test $\sum f_n'(x)$ convergent uniformly

By Corollary 5.3.4 $\quad \left(\sum_{k=1}^{\infty} \frac{\cos(2^k x)}{3^k}\right)' = \sum_{k=1}^{\infty} f_n'(x)$ $\quad$ [illegible]

[illegible]

$\because \sum_{k=1}^{\infty} f_n'$ convergent uniformly and $f_n = \frac{-\sin(5x)^2}{3^k}$ is continuous $\forall n \in \mathbb{N}$

By 1-1.5 $\sum_{k=1}^{\infty} f_n'$ is continuous

f. By Taylor's Theorem

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^{99}}{99!} + R$

Check $|e^x - 1| \leq e^x x$ $\quad \because e^x$ is monotone increasing

Pf: $|e^x - 1| = \int_0^x e^t \, dt = e^x \int_0^x 1 \, dt = e^x x$

$\because e^x > 1 \quad > 0$

Check $|e^x - 1 - x| \leq \frac{e^x x^2}{2}$ $\quad e^t - 1 > 0$

$$|e^x - 1 - x| = \left|\int_0^x e^t - 1 \, dt\right| = \int_0^x e^t - 1 \, dt \leq \int_0^x e^t - 1 \, dt$$

$$\leq \int_0^x e^t t \, dt$$

$e^x \uparrow \quad \leq e^x \int_0^x t \, dt = e^x \frac{x^2}{2}$

If $\left|e^x - 1 - x - \frac{x^2}{2} - \cdots - \frac{x^{98}}{98!}\right| \leq e^x \frac{x^{99}}{99!}$

$$\left|e^x - 1 - x - \frac{x^2}{2} \cdots - \frac{x^{98}}{98!} - \frac{x^{99}}{99}\right| \leq \left|\int_0^x e^t - 1 - t - \frac{t^{98}}{98!} \, dt\right|$$

$$\leq \int_0^x \left|e^t - 1 - t - \frac{t^{98}}{98!}\right| dt$$

$$\leq \int_0^x \frac{e^t t^{99}}{99!} dt \qquad x > 0$$

$$\leq e^x \int_0^x \frac{t^{99}}{99} dt = e^x \frac{x^{100}}{100!}$$

g. $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}$

$\because \left|\frac{x^n}{n^3}\right| \leq \frac{1}{n^3} \quad x \in [-1, 1]$

By M-test $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ convergent uniformly

Let $f_n(x) = \frac{x^n}{n^3} \quad x \in [-1, 1] \Rightarrow f_n(x)$ is continuous By 1-1.5 $f(x)$ is continuous h $\lim_{x\to 0} \frac{e^{ax}-1}{\sin bx} = \lim_{x\to 0} \frac{a e^{ax}}{b\cos bx} = \frac{a}{b}$

l'Hopital's rule