3.

$$\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{(x_0,y_0)} = \begin{vmatrix} f'(x_0) & 0 \\ x_0 f'(x_0)+f(x_0) & -1 \end{vmatrix} = -f'(x_0) \neq 0.$$

By inverse function theorem,

the transformation $(x,y) \longmapsto (u(x,y), v(x,y))$ is invertible near (x_0, y_0).

Since $u=f(x)$, we have

$$x = f^{-1}(u), \quad \text{and} \quad y = -v + xf(x) = -v + uf^{-1}(u).$$

#

6.

Let $F_1(x,y) = x^2+y+\sin(xy)$. Then

$$\frac{\partial F_1}{\partial x} = 2x + y\cos(xy) \qquad \Rightarrow \quad \frac{\partial F_1}{\partial x}(0,0) = 0$$

$$\frac{\partial F_1}{\partial y} = 1 + x\cos(xy) \qquad \quad \frac{\partial F_1}{\partial y}(0,0) = 1$$

Thus near $(0,0)$, $F_1 = 0$ can be written as $y = f(x)$, can not be written as $x = h(y)$.

11.

a. The rank of $Df(x_0) = m \Rightarrow m \leq n$.

Without loss of generality, we may assume that $\left.\dfrac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}\right|_{(x_0)} \neq 0$

Let $F(x) = (f_1, f_2, \dots, f_m, x_{m+1}, \dots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}^m \times \mathbb{R}^{n-m}$.

Then
$$JF(x_0) = \left|\begin{matrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} & \frac{\partial f_1}{\partial x_{m+1}} & \frac{\partial f_1}{\partial x_{m+2}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial z} & \cdots & \frac{\partial f_2}{\partial x_m} & \frac{\partial f_2}{\partial x_{m+1}} & \frac{\partial f_2}{\partial x_{m+2}} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & & \frac{\partial f_m}{\partial x_m} & \frac{\partial f_m}{\partial x_{m+1}} & \frac{\partial f_m}{\partial x_{m+2}} & \cdots & \frac{\partial f_m}{\partial x_n} \\ 0 & 0 & & 0 & 1 & 0 & & 0 \\ 0 & 0 & & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 & 0 & 0 & \cdots & 0 \\ & & & & & & & 1 \end{matrix}\right|_{(x_0)} = \left|\begin{matrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & & \frac{\partial f_m}{\partial x_m} \end{matrix}\right|_{(x_0)}$$
$$= \left.\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}\right|_{(x_0)} \neq 0$$

By inverse function thm, $\exists$ open nbh A of x_0, U of $F(x_0)$

s.t. $F(A) = U$ and $\exists$ a inverse $F^{-1} : U \to A$

Thus, $F(x_0) = (f_1(x_0), \dots, f_m(x_0), x_{m+1_0}, \dots, x_{n_0}) \in U$

$\exists$ a whole neighborhood W of $f(x_0)$

s.t. $\forall y \in W$

$F^{-1}(y, x_{m+1_0}, \dots, x_{n_0}) = (x_1, \dots, x_m, x_{m+1_0}, \dots, x_{n_0})$.

$\Rightarrow f(x_1, \dots, x_m, x_{m+1_0}, \dots, x_{n_0}) = y$

i.e, there is a whole nbh of $f(x_0)$ lying in the image of f. ※

#11. b.

$\because$ $Df(x_0)$ is 1-1 $\Rightarrow$ rank $Df(x_0) = n$ and $n \le m$.

We may assume $\left.\dfrac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}\right|_{x=x_0} \neq 0.$

$\because$ $f \in C^1$ $\Rightarrow$ $\exists$ a nbh $D(x_0,r)$

s.t.

$$(*) \quad \begin{vmatrix} \frac{\partial f_1(z_1)}{\partial x_1} & \frac{\partial f_1(z_1)}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}(z_1) \\ \frac{\partial f_2(z_2)}{\partial x_1} & \frac{\partial f_2(z_2)}{\partial x_2} & \cdots & \frac{\partial f_2(z_2)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(z_n)}{\partial x_1} & \frac{\partial f_n(z_n)}{\partial x_2} & \cdots & \frac{\partial f_n(z_n)}{\partial x_n} \end{vmatrix} \neq 0 \quad \forall\ z_1, z_2, \cdots, z_n \in D(x_0,r)$$

If $x \neq y$, $x,y \in D(x_0,r)$, then by M.V.T.

$$\begin{cases} f_1(y) - f_1(x) = Df_1(z_1)(y-x) \\ f_2(y) - f_2(x) = Df_2(z_2)(y-x) \\ \quad\vdots \\ f_n(y) - f_n(x) = Df_n(z_n)(y-x) \end{cases}$$

$\because$ $Df_1(z_1), Df_2(z_2), \cdots, Df_n(z_n)$ are linearly independent by $(*)$,

$\therefore$ $(f_1(y)-f_1(x), f_2(y)-f_2(x), \cdots, f_n(y)-f_n(x)) \neq (0,0,\cdots,0)$.

$\Rightarrow$ f is 1-1 on $D(x_0,r)$ #.

12.

Suppose $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^1 and $Jf(x_0) \neq 0$.

Let $F: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $F(x, y) = f(x) - y$.

Then
$$\left.\frac{\partial(F_1, F_2, \cdots, F_n)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x=x_0} = Jf(x_0) \neq 0,$$

by implicit function theorem, for $F(x_0, y_0) = 0$

$\exists$ an open nbh $U \subset \mathbb{R}^n$ of x_0, an open nbh V of y_0 in $\mathbb{R}^n$ and a unique function $g: V \longrightarrow U$ such that

$$F(g(y), y) = 0 \quad \forall\, y \in V$$

Thus, $x = g(y) \quad \forall\, x \in U,\ y \in V$

i.e. for $f(x) = y \Rightarrow x = g(y) \quad \forall\, x \in U,\ y \in V$.

$\Rightarrow g = f^{-1}$ on V #.

\# 25. $B(0,r) = \{ x \in \mathbb{R}^n \mid \|x\| \leq r \}$,

Claim : $f(x) \in B(0,r) \quad \forall x \in B(0,r)$.

Pf: $\|f(x)\| \leq \|f(x) - f(0)\| + \|f(0)\|$

$\leq \frac{1}{3}\|x\| + \frac{2}{3}r$

$\leq \frac{1}{3}r + \frac{2}{3}r = r \quad \forall x \in B(0,r)$

$\Rightarrow f(x) \in B(0,r)$

Thus $f : B(0,r) \longrightarrow B(0,r)$

Since $\|f(x) - f(y)\| \leq \frac{1}{3}\|x-y\| \quad \forall x, y \in B(0,r)$

$\Rightarrow$ f is a contraction mapping on $B(0,r)$

Since $B(0,r)$ is a complete metric space and f is a contraction mapping on $B(0,r)$,

f has a unique fixed point $x \in B(0,r)$.

That is $\exists!\ x \in B(0,r)$ s.t. $f(x) = x$ #