Exercise for § 6.2

3 Let L be a linear map of $\mathbb{R}^n \to \mathbb{R}^m$, Let $g: \mathbb{R}^n \to \mathbb{R}^m$ be such that $\|g(x)\| \le M\|x\|^2$, and let $f(x) = L(x) + g(x)$. Prove that $Df(0) = L$

pf: By § 6.1 exercise 4 $\Rightarrow$ $Dg(0) = 0$

By 6.2.4 example $DL = L$

$Df(0) = DL(0) + Dg(0) = L + 0 = L$

4 Discuss the possibility of defining Df for f a mapping from one normed space to another

Ans Let $f: A \subset M \to N$ $\quad$ M, N are norm space

f is said to be differentiable at $x \in A$ if there is a linear function denoted $Df(x_0) = M \to N$ and called the derivative of f at x_0, such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_N}{\|x - x_0\|_M} = 0$$

If $M = \mathbb{R}^n$ $N = \mathbb{R}^m$

$Df(x_0)$ is continuous

But M, N are infinite dimensional space

$Df(x_0)$ maybe not continuous

Exercise for § 6.3

1 Let $f(x)=x^2$ if x is irrational and let $f(x)=0$ if x is rational. Is f continuous at 0?

Is it differentiable at 0?

pf = Check f is continuous at 0

$\forall \varepsilon>0$ Let $\delta=\sqrt{\varepsilon}$ $|f(x)-f(0)|=|x^2|<\varepsilon$ if $|x|<\delta$

check f is differentiable at 0

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2}{x}=0 \quad \text{if } x\in\mathbb{Q}^c$$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0 \quad \text{if } x\in\mathbb{Q}$$

$\Rightarrow$ f is differentiable at 0 and $Df(0)=0$

2 Is the local Lipschitz condition in Theorem 6.3.1 enough to guarantee differentiablity

Ans No Let $f(x)=|x|$ $x\in\mathbb{R}$

$|f(x)-f(y)|\le 1\,|x-y|$

f not differentiable at 0

Exercise for §6.8

3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{if } x \in (-1\ \ 1) \quad x \neq 0$$

and

$$f(x) = 0 \quad \text{if } x = 0$$

Investigate the validity of Taylor's theorem for f about the point $x=0$

Ans:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 \quad \left(\because \left|\sin\left(\frac{1}{x}\right)\right| \le 1\right)$$

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\frac{1}{x}$$

$\lim_{x \to 0} f'(x)$ does not exist ($\because \cos\frac{1}{x}$ oscillates between $+1$ and -1)

$\Rightarrow f'(x)$ is not continuous at 0

5 Compute the second-order Taylor formula for $f(x,y) = e^x \cos y$ around $(0,0)$

Ans:

$$\frac{\partial f}{\partial x} = e^x \cos y \qquad f(0,0) = 1$$

$$\frac{\partial f}{\partial y} = -e^x \sin y \qquad \frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \cos y \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = -1$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -e^x \sin y$$

$$f(h,k) = 1 + (1,0)\cdot(h,k) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \cdot (h,k) + R_2(h,k,0)$$

$$= 1 + h + \frac{1}{2}(h^2 - k^2) + R_2((h,k),0)$$

where $R_2((h,k),0) / \|(h,k)\|^2 \to 0$ as $(h,k) \to (0,0)$

Exercise for § 6.9

1 Prove that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

is negative definite iff $a<0$ and $ad-b^2>0$

Pf: "$\Rightarrow$" Negative definite means

$$(x \cdot y)\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} < 0 \quad \text{if} \quad (x,y) \neq (0,0)$$

$$\Leftrightarrow ax^2+2bxy+dy^2<0$$

Let $(x,y)=(1,0) \Rightarrow a<0$

Let $y=1$ $\quad ax^2+2bx+d<0$ for all x

The function has maximum at $2ax+2b=0$

$$\Rightarrow x=-b/a$$

$$\Rightarrow a\left(-\frac{b}{a}\right)^2+2b\left(-\frac{b}{a}\right)+d<0$$

$$\frac{b^2}{a}-\frac{2b^2}{a}+d<0$$

$$ab-b^2<0$$

"$\Leftarrow$" same way

4 (This exercise assumes a knowledge of linear algebra) Let A be a symmetric matrix. Show that A is positive definite if and only if the eigenvalues of A (which exist and are real, since A is symmetric) are positive. Is this true if A is not symmetric

(1)

Pf: $\because$ A is symmetric

$A = U^{*}\lambda U \qquad U^{*}U = UU^{*} = I$

$\lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\langle Ax, x\rangle = \langle U^{*}\lambda U x, x\rangle$

$= \langle \lambda U x, Ux\rangle > 0$ if $\lambda_1 \cdots \lambda_n > 0$ $\quad x \neq 0$

$\Rightarrow$ A is positive definite

(2) $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $\qquad \langle A(1,1), (1,1)\rangle = -1$

$[1 \ 1]\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1$

$(1,1)\cdot(1 \ -2)$

Check that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has $\Delta_i \geq 0$ yet the matrix is not semidefinite

Ans Let $e_1 = (1.0.0)$ $e_3 = (0.0.1)$

Then $\langle Ae_1, e_1\rangle = 1$

but $\langle Ae_3 \ e_3\rangle = -1$ so A is not semidefinite