Exercises for Chapter 7

15. Consider the map $\mathcal{L}^{-1} : GL(\mathbb{R}^n, \mathbb{R}^n) \to GL(\mathbb{R}^n, \mathbb{R}^n)$, $A \mapsto A^{-1}$, taking a matrix to its inverse. show that the derivative of this map is given by

$$D\mathcal{L}^{-1}(A) \cdot B = -A^{-1} \circ B \circ A^{-1} \,.$$

Proof. The goal is to show that

$$\lim_{\|h\| \to 0} \frac{\big\|(A+h)^{-1} - A^{-1} - (-A^{-1}hA^{-1})\big\|}{\|h\|} = 0 \,,$$

where $\|h\| = \sup\limits_{\|x\|_{\mathbb{R}^n}=1} \|hx\|_{\mathbb{R}^n}$. We first note that

$$(A+h)^{-1} - A^{-1} = A^{-1}A(A+h)^{-1} - A^{-1}(A+h)(A+h)^{-1} = -A^{-1}h(A+h)^{-1} \,;$$

thus

$$(A+h)^{-1} - A^{-1} - (-A^{-1}hA^{-1}) = A^{-1}h\big[A^{-1} - (A+h)^{-1}\big] = A^{-1}hA^{-1}h(A+h)^{-1} \,.$$

Therefore, by that $\|AB\| \le \|A\|\|B\|$ for all $A, B \in GL(\mathbb{R}^n, \mathbb{R}^n)$, we find that

$$\frac{\big\|(A+h)^{-1} - A^{-1} - (-A^{-1}hA^{-1})\big\|}{\|h\|} \le \|A^{-1}\|^2 \|(A+h)^{-1}\| \|h\| \to 0$$

as $\|h\| \to 0$. $\square$