Exercises within a Section: 1.1-2 means Section 1.1 problem #2 etc...

☞**Watch out! To save paper and spaces, some solutions may not be in the proper order. You should be able to find them.**（解答裡題目的順序可能不會照書上的順序，請大家在「該出現的地方」找不到時往後翻一下）

☞**You are required to reproduce or to paraphrase of the "Solution" (NOT the "Sketch") to a problem.** （考試寫證明的時候，是要寫 Solution 的部份，而不是 Sketch 的部份！Sketch 的部份是告訴大家證明的 idea 是什麼。）

☞**Do not skip over problems that you think are complicated. We can still ask you part of the steps in a test.**（不要跳過你覺得證明太複雜的問題，我們考試時仍可能會考你！）

◇ **4.1-1.** (a) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Prove that f is continuous.
(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. Prove that f is continuous.

Suggestion. For $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, try $\delta = \min(1, \varepsilon/(1 + 2|x_0|))$. ◇

Solution. (a) To find the form of the solution, compute the quantity which is to be made small. For $x_0 \in \mathbb{R}$ we have

$$|f(x) - f(x_0)| = \left|x^2 - x_0^2\right| = |x + x_0|\,|x - x_0|\,.$$

If $|x - x_0| < \delta$, then $|x + x_0| \le |x - x_0| + |2x_0| \le \delta + 2|x_0|$. If $\delta < 1$, this leaves us with $|f(x) - f(x_0)| < (1 + 2|x_0|)\,|x - x_0|$. So, if we take δ to be the smaller of the two numbers 1 and $\varepsilon/((1 + 2|x_0|)$, we have

$$\begin{aligned}|f(x) - f(x_0)| &\le (\delta + 2|x_0|)\,|x - x_0| \le (1 + 2|x_0|)\,|x - x_0| \\ &\le (1 + 2|x_0|)\frac{\varepsilon}{1 + 2|x_0|} = \varepsilon.\end{aligned}$$

Thus f is continuous at x_0. Since x_0 was an arbitrary point in $\mathbb{R}$, we conclude that f is continuous on all of $\mathbb{R}$.

(b) Solution One: If $v = (a, b) \in \mathbb{R}$, then $f(v) = a$. This is called *projection onto the first coordinate*. If (x, y) is another point, then

$$|f(x, y) - f(a, b)| = |x - a| \le \sqrt{(x - a)^2 + (y - b)^2} = \|\,(x, y) - (a, b)\,\|\,.$$

Let $\varepsilon > 0$. If we take $\delta = \varepsilon$, we find that if $\|\,(x, y) - (a, b)\,\| < \delta$, then

$$|f(x, y) - f(a, b)| \le \|\,(x, y) - (a, b)\,\| < \delta = \varepsilon.$$

Thus f is continuous at (a, b). Since (a, b) was an arbitrary point in $\mathbb{R}^2$, we conclude that f is continuous on $\mathbb{R}^2$.

Solution Two: Another way to handle this exercise is to use the characterization of continuity in terms of convergent sequences given in Theorem 4.1.4(ii) together with facts we already know about the convergence of sequences in $\mathbb{R}^2$. Let $(x_k, y_k) \to (a, b))$ in $\mathbb{R}^2$. We know from Chapter 2 that this happens (with respect to the usual Euclidean distance in $\mathbb{R}^2$) if and only if $x_k \to a$ and $y_k \to b$ in $\mathbb{R}$. In particular, $f((x_k, y_k)) = x_k \to a$. Since this happens for every sequence in $\mathbb{R}^2$ converging to (a, b), Theorem 4.1.4(ii) says that f is continuous at (a, b). Since (a, b) was arbitrary in $\mathbb{R}^2$, we conclude that f is continuous on $\mathbb{R}^2$.

The function studied in part (b) is called the projection of $\mathbb{R}^2$ onto the first coordinate. It is sometimes denoted by π_1. The projection onto the second coordinate is defined similarly: $\pi_2((x, y)) = y$. This notation might or might not be a good idea, but it is fairly common. It is one of the very few times when it is permissible to use the symbol "π" for something other than the ratio of the circumference of a circle to its diameter.

For related material, see Exercise 3E-15. ♦

◇ **4.1-5.** Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $U \subset \mathbb{R}$ such that $f(U)$ is *not* open.

Sketch. $f(x) = 1$, U = any open set;

$$f(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \ge 1. \end{cases} \quad U =]-1, 2[;\ f(U) = [0, 1] \text{ is closed.}$$ ◇

Solution. For a very easy example, take the constant function $f(x) = 1$ for all $x \in \mathbb{R}$. If $a \in \mathbb{R}$ and $x_k \to a$, then $f(x_k) = f(a) = 0$ for every k. So we certainly have $f(x_k) \to f(a)$. Thus f is continuous at a by 4.1.4(ii), and, since a was an arbitrary point in $\mathbb{R}$, f is continuous on $\mathbb{R}$. If U is any nonempty open subset of $\mathbb{R}$ then $f(U) = \{1\}$. This one point set is not an open subset of $\mathbb{R}$.

For a slightly more imaginative example, we could take

$$f(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Then if $U =]-1, 2[$, we have $f(U) = [0, 1]$, which is closed. (Show f is continuous.) ♦

◇ **4.2-1.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Which of the following sets are necessarily closed, open, compact, or connected?

(a) $\{x \in \mathbb{R} \mid f(x) = 0\}$

(b) $\{x \in \mathbb{R} \mid f(x) > 1\}$

(c) $\{f(x) \in \mathbb{R} \mid x \ge 0\}$

(d) $\{f(x) \in \mathbb{R} \mid 0 \le x \le 1\}$

Answer. (a) Closed, not necessarily compact or connected.

(b) Open, not necessarily compact or connected.

(c) Connected, not necessarily compact, open, or closed.

(d) Compact, closed, and connected; not necessarily open. ◇

Solution. (a) If $A = \{x \in \mathbb{R} \mid f(x) = 0\}$, then $A = f^{-1}(\{0\})$. Since the one point set $\{0\}$ is closed in $\mathbb{R}$, and f is continuous, A must be closed by Theorem 4.1.4(iv).

If we use the function $f(x) = 0$ for all x, then f is continuous on all of $\mathbb{R}$. One way to see this is to let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. If $x \in \mathbb{R}$, we have $|f(x) - f(x_0)| = |0 - 0| = 0 < \varepsilon$. So we can let δ be any positive number to get $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. Thus f is continuous at x_0, and, since x_0 was arbitrary in $\mathbb{R}$, f is continuous on $\mathbb{R}$. For this function we have $f^{-1}(\{0\}) = \mathbb{R}$ which is not bounded and not compact. So A need not be compact.

Now consider $f(x) = x^2 - 1$. For this function we have $A = f^{-1}(\{0\}) = \{\pm 1\}$, a two point set which is not connected. To see that f is continuous, note that $|f(x) - f(x_0)| = |x^2 - x_0^2|$. Given $\varepsilon > 0$, select $\delta > 0$ as in Exercise 4.1-1(a). Thus A need not be connected.

(b) If $B = \{x \in \mathbb{R} \mid f(x) > 1\}$, then $B = f^{-1}(U)$ where $U = \{y \in \mathbb{R} \mid y > 1\}$. Since U is open in $\mathbb{R}$ and f is continuous, B must be open by Theorem 4.1.4(iii).

To see that B need not be compact, consider the constant function $f(x) = 2$ for all $x \in \mathbb{R}$. Modify the argument in part (a) slightly to show that f is continuous on $\mathbb{R}$. Since $2 > 1$, we have $B = f^{-1}(U) = \mathbb{R}$ which is not compact.

To see that B need not be connected, let $f(x) = x^2 - 1$. Then f is continuous as in part (a). $f(x) > 1 \iff |x| > \sqrt{2}$, so $B = \{x \in \mathbb{R} \mid x < -\sqrt{2}\} \cup \{x \in \mathbb{R} \mid x > \sqrt{2}\}$. Since B is the union of two disjoint, nonempty open sets, it is not connected.

(c) If $C = \{f(x) \in \mathbb{R} \mid x \geq 0\}$, then $C = f(J)$ where J is the closed half line $\{x \in \mathbb{R} \mid x \geq 0\}$. Since J is path-connected it is connected. Since f is continuous on J, $C = f(J)$ must be connected by Theorem 4.2.1.

To see that $f(J)$ need not be open, consider the function $f(x) = 0$ for all x used in part (a). We know that f is continuous, and $C = f(J) = \{0\}$. This one point set is not open in $\mathbb{R}$.

To see that C need not be closed, consider the function $f(x) = 1/(x^2+1)$ for all $x \in \mathbb{R}$. The arithmetic of limits allows us to conclude from 4.1.4(ii) that f is continuous. If $x_k \to a$, then $x_k^2 \to a^2$, and $x_k^2 + 1 \to a^2 + 1 \geq 1$. So $f(x_k) = 1/(x_k^2 + 1) \to 1/(a^2 + 1) = f(a)$. This shows that f is continuous at a, and, since a was an arbitrary point in $\mathbb{R}$, that f is continuous on $\mathbb{R}$. If $x \in \mathbb{R}$, then $1 \leq x^2 + 1 < \infty$, so $0 < f(x) \leq 1$. On the other hand, if $0 < y \leq 1$, then we can put $x = \sqrt{(1-y)/y}$ and compute that $f(x) = y$. So $f(J)$ is the half-open interval $]0, 1]$. This is not closed since $0 \in \mathrm{cl}(]0,1]) \setminus]0,1]$. This supplies the required counterexample.

Since the half-open interval $]0, 1]$ is not closed, it is not compact, and the same example as in the last paragraph shows that C need not be compact.

(d) If $D = \{f(x) \in \mathbb{R} \mid 0 \leq x \leq 1\}$, then $D = f(K)$ where K is the closed unit interval $[0, 1]$. Since K is compact, the image $D = f(K)$ must be compact by Theorem 4.2.2. Since it is a compact subset of $\mathbb{R}$, the set D must be closed. Since the interval K is connected, the image $D = f(K)$ must be connected by Theorem 4.2.1.

To see that D need not be open, use the function $f(x) = x$ for all x. Then f is continuous by Example 4.1.5, and $D = f(K) = K = [0, 1]$. The image D is not open since $0 \in D$ but no small interval around 0 is contained in D. ♦

◇ **4.2-3.** Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ and a closed subset $B \subset \mathbb{R}$ such that $f(B)$ is not closed. Is this possible if B is bounded as well?

Answer. It is not possible if B is closed and bounded since then it is compact. ◇

Solution. An example is given in part (c) of Exercise 4.2-1. $B = \{x \in \mathbb{R} \mid x \geq 0\}$, and $f(x) = 1/(x^2 + 1)$. Then $f(B) =]0, 1]$ which is not closed.

If B is a subset of $\mathbb{R}$, which is both closed and bounded, then it is compact. Theorem 4.2.2 says that $f(B)$ would also be compact. It would thus be a closed subset of $\mathbb{R}$. ♦

◇ **4.2-5.** Let A and B be subsets of $\mathbb{R}$ with B not empty. If $A \times B \subseteq \mathbb{R}^2$ is open, must A be open?

Answer. Yes. ◇

Solution. Since B is not empty, there is a point $b \in B$. If $a \in A$, then $(a,b) \in A \times B$. Since $A \times B$ is open, there is an $r > 0$ such that $\sqrt{(x-a)^2 + (y-b)^2} < r$ implies $(x,y) \in A \times B$. If $|x-a| < r$, then $\sqrt{(x-a)^2 + (b-b)^2} = |x-a| < r$, so $(x,b) \in A \times B$. Thus $x \in A$. For each $a \in A$ there is an $r > 0$ such that $x \in A$ whenever $|x-a| < r$. So A is open. ♦

◇ **4.3-1.** Where are the following functions continuous?

(a) $f(x) = x\sin(x^2)$.
(b) $f(x) = (x + x^2)/(x^2 - 1)$, $x^2 \neq 1$, $f(\pm 1) = 0$.
(c) $f(x) = (\sin x)/x$, $x \neq 0$, $f(0) = 1$.

Answer. (a) Everywhere.
(b) f is continuous on $\mathbb{R} \setminus \{-1, 1\}$.
(c) Everywhere. ◇

Solution. (a) If $f(x) = x\sin(x^2)$, then f is the product of the continuous function $x \mapsto x$ (Example 4.1.5) with the composition $x \mapsto \sin(x^2)$. The function $x \mapsto x^2$ is continuous. This was seen directly in Exercise 4.1-1(a). We now have an easier indirect proof since it is the product of $x \mapsto x$ with itself. The final fact we need is that $\vartheta \mapsto \sin\vartheta$ is continuous. How this is proved depends on just how the sine function is defined. We will just assume it here. Since products and compositions of continuous functions are continuous, this function is continuous everywhere.

(b) The numerator and denominator of $f(x) = (x + x^2)/(x^2 - 1)$ are continuous everywhere since products and sums of continuous functions are continuous. Since quotients are continuous except where the denominator is 0, the only possible discontinuities are at 1 and -1. The limit at 1 does not exist, and the limit at -1 is $1/2$ since

$$\lim_{x \to -1} \frac{x + x^2}{x^2 - 1} = \lim_{x \to -1} \frac{x(x+1)}{(x+1)(x-1)} = \lim_{x \to -1} \frac{x}{x-1} = \frac{-1}{-1-1} = \frac{1}{2}.$$

Since $1/2 \neq 0 = f(-1)$, f is not continuous at -1. Thus f is continuous on $\mathbb{R} \setminus \{\pm 1\}$.

(c) Again we assume the the sine function is continuous everywhere. So the numerator and denominator of $f(x) = (\sin x)/x$ are continuous everywhere. The only possible discontinuity is at $x = 0$ where the denominator is 0. But the numerator is also 0 there. We know from L'Hôpital's Rule that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1 = f(0).$$

So f is continuous at 0. It is thus continuous everywhere. ♦

◇ **4.3-3.** Let $A = \{x \in \mathbb{R} \mid \sin x = 0.56\}$. Show that A is a closed set. Is it compact?

Suggestion. Use the fact that $\{0.56\}$ is closed and $\sin x$ is continuous. A is not compact. ♢

Solution. Again we assume that the function $f(x) = \sin x$ is continuous. Then $A = \{x \in \mathbb{R} \mid \sin x = 0.56\} = f^{-1}(\{0.56\})$. Since the one point set

$\{0.56\}$ is closed in $\mathbb{R}$ and f is continuous, A must also be closed. It is not compact since if x_0 is any one point in A, then $x_0 + 2\pi k$ is also in A for every integer k. Thus A is not bounded and cannot be compact. ♦

◇ **4.4-1.** Give an example of a continuous and bounded function on all of $\mathbb{R}$ that does not attain its maximum or minimum.

Answer. One possibility is $f(x) = x/(1 + |x|)$. ♢

Solution. If we let $f(x) = x/(1 + |x|)$ for all $x \in \mathbb{R}$, then the numerator and denominator are continuous everywhere and the denominator is never 0, so f is continuous everywhere on $\mathbb{R}$. Furthermore, $|f(x)| = |x|/(1+|x|) < 1$. So $-1 < f(x) < 1$ for all $x \in \mathbb{R}$. Finally

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{1+x} = \lim_{x\to+\infty} \frac{1}{1+(1/x)} = \frac{1}{1+0} = 1$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{1-x} = \lim_{x\to+\infty} \frac{1}{-1+(1/x)} = \frac{1}{-1+0} = -1$$

So $\sup\{f(x) \mid x \in \mathbb{R}\} = 1$ and $\inf\{f(x) \mid x \in \mathbb{R}\} = -1$ and neither of these is attained anywhere in $\mathbb{R}$. See Figure 4-3. ♦

◇ **4.4-5.** Is a version of the maximum-minimum theorem valid for the function $f(x) = (\sin x)/x$ on $]0, \infty[$? On $[0, \infty[$?

Sketch. $\sup(f(]0, \infty[)) = 1$ is not attained on $]0, \infty[$. Extend by $f(0) = 1$ (continuous?) to get it on $[0, \infty[$. ♢

Solution. If $f(x) = (\sin x)/x$ for x not 0, then we know from calculus or elsewhere that $\lim_{x\to 0} f(x) = 1$. So if we define $f(0)$ to be 0, we obtain a continuous function on $\mathbb{R}$. $f'(x) = (x\cos x - \sin x)/x^2$. This is 0 only when $x = \tan x$. This occurs at $x = 0$ in the limit and not again until $|x| > \pi$. See Figure 4-5.

We have $f(\pm\pi) = 0$, $f'(x) > 0$ for $-\pi < x < 0$, and $f'(x) < 0$ for $0 < x < \pi$. Consequently, $0 < f(x) < \lim_{x\to 0} f(x) = 1 = f(0)$ for $0 < x < \pi$. For $|x| > \pi$, we have $|f(x)| = |\sin x|/|x| < 1/|x| < 1/\pi$. Thus $\sup\{f(x) \mid x \in \mathbb{R}\} = 1 = f(1)$ and $\inf\{f(x) \mid x \in \mathbb{R}\}$ occurs at the two points with $\pi/2 < |x| < \pi$ at which the derivative is 0. The supremum is attained on $[0, \infty[$, but not on $]0, \infty[$. See Figure 4-6.

What is going on might be summarized by something like the following.

Proposition. *If f is a continuous real valued function on the closed half line $[a, \infty[$ and*

$$\limsup_{x\to\infty} f(x) < \sup\{f(x) \mid x \in [a, \infty[\,\} < \infty,$$

then there is at least one point $x_1 \in [a, \infty[$ at which $f(x_1) = \sup\{f(x) \mid x \in [a, \infty[\,\}$. If

$$\liminf_{x\to\infty} f(x) > \inf\{f(x) \mid x \in [a, \infty[\,\} > -\infty,$$

then there is at least one point $x_2 \in [a, \infty[$ at which $f(x_2) = \inf\{f(x) \mid x \in [a, \infty[\,\}$.

The hypotheses say that there is a number $B > a$ such that $f(x)$ is smaller than $\sup\{f(x) \mid x \in [a, \infty[\,\}$ and larger than $\inf\{f(x) \mid x \in [a, \infty[\,\}$ for $x > B$. The supremum and infimum over $[a, \infty[$ are thus the same as those over the compact interval $[a, B]$. We can use the maximum-minimum theorem to conclude that they are attained at points in that interval. ♦

◇ **4.5-3.** Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point.

Suggestion. Apply the intermediate value theorem to $g(x) = f(x) - x$. ◇

Solution. For $x \in [0, 1]$, let $g(x) = f(x) - x$. Since f is continuous and $x \mapsto -x$ is continuous, g is continuous. We have

$$g(0) = f(0) - 0 = f(0) \geq 0 \quad \text{and} \quad g(1) = f(1) - 1 \leq 1 - 1 = 0$$

By the intermediate value theorem, there must be at least one point c in $[0, 1]$ with $g(c) = 0$. For such a point we have $f(c) - c = 0$. So $f(c) = c$. Thus c is a fixed point for the mapping f as required.

See Figure 4-7. ♦

◇ **4.5-5.** Prove that there is no continuous map taking $[0, 1]$ onto $]0, 1[$.

Sketch. $f([0, 1])$ would be compact, and $]0, 1[$ is not compact. ◇

Solution. The interval $[0, 1]$ is a closed bounded subset of $\mathbb{R}$ and so it is compact. If f were a continuous function on it, then the image would have to be compact. The open interval $]0, 1[$ is not compact. So it cannot be the image of such a map. ♦

◇ **4.6-2.** Prove that $f(x) = 1/x$ is uniformly continuous on $[a, \infty[$ for $a > 0$.

Suggestion. Compare to Exercise 4.6-1. ◇

Solution. We are asked to show that the function $f(x) = 1/x$ is uniformly continuous on the half line $[a, \infty[$ if $0 < a$. To do this we compute

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{|xy|} \leq \frac{|x - y|}{a^2}.$$

If $\varepsilon > 0$, let $\delta = a^2\varepsilon$. If x and y are in $[a, \infty[$ and $|x - y| < \delta$, then the last computation shows that

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| =\leq \frac{|x - y|}{a^2} < \frac{a^2\varepsilon}{a^2} = \varepsilon.$$

The same δ works everywhere on $[a, \infty[$, so f is uniformly continuous on that domain. ♦

◇ **4.6-1.** Demonstrate the conclusion in Example 4.6.3 directly from the definition.

Sketch. $|(1/x) - (1/y)| = |(x-y)/xy| \leq |x-y|/a^2$. Take $\delta = a^2\varepsilon$. ◇

Solution. We are asked to show that the function $f(x) = 1/x$ is uniformly continuous on the interval $[a,1]$ if $0 < a < 1$. To do this we compute

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|xy|} \leq \frac{|x-y|}{a^2}.$$

If $\varepsilon > 0$, let $\delta = a^2\varepsilon$. If x and y are in $[a,1]$ and $|x-y| < \delta$, then the last computation shows that

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \leq \frac{|x-y|}{a^2} < \frac{a^2\varepsilon}{a^2} = \varepsilon.$$

The same δ works everywhere on $[a,1]$, so f is uniformly continuous on that domain. ♦

◇ **4.6-3.** Must a bounded continuous function on $\mathbb{R}$ be uniformly continuous?

Sketch. No. Consider $f(x) = \sin(x^2)$. ◇

Solution. To get a bounded continuous function which is not uniformly continuous, we want a function whose derivative stays large over nontrivial regions. If we put $f(x) = \sin(x^2)$, then $f'(x) = 2x\cos(x^2)$. So f has a large derivative near $x = \sqrt{2\pi n}$ for integer n. See Figure 4-8.

The function f is continuous. To show that it is not uniformly continuous, we take $\varepsilon = 1/2$, and show that for any $\delta > 0$ there are points x and y

with $|x-y| < \delta$ and $|f(x) - f(y)| > 1/2$. To this end let $x = \sqrt{2\pi n}$ and $y = \sqrt{2\pi n + (\pi/2)}$. Then

$$y - x = \sqrt{2\pi n + \frac{\pi}{2}} - \sqrt{2\pi n} = \frac{(2\pi n + \frac{\pi}{2}) - 2\pi n}{\sqrt{2\pi n + \frac{\pi}{2}} + \sqrt{2\pi n}} < \frac{\pi/2}{2\sqrt{2\pi n}}.$$

Since this tends to 0 as n increases, we can pick n large enough so that $|y-x| < \delta$. But $|f(y) - f(x)| = |\sin(2\pi n) - \sin(2\pi n + (\pi/2))| = |0 - 1| = 1$. So no choice of $\delta > 0$ can work everywhere in $\mathbb{R}$. So f is not uniformly continuous on $\mathbb{R}$. ♦

◇ **4.7-2.** Does the mean value theorem apply to $f(x) = \sqrt{x}$ on $[0,1]$? Does it apply to $g(x) = \sqrt{|x|}$ on $[-1,1]$?

Solution. The function $f(x) = \sqrt{x}$ is continuous on the closed interval $[0,1]$. (See Exercise 4.3-4.) and differentiable on the interior $[0,1]$. (To see this, you can use the Inverse Function Theorem 4.7.15. f is the inverse of the differentiable function $x \mapsto x^2$.) So the mean value theorem does apply.

The function $g(x) = \sqrt{|x|}$ is continuous on $[-1,1]$. (Again, see Exercise 4.3-4.) But it is not differentiable at the point 0 in the interior of that interval. So the mean value theorem does not directly apply. The theorem can be applied separately on the intervals $[-1,0]$ and $[0,1]$. With some care about how the function joins together at 0, the same conclusion can be drawn even though the theorem cannot be applied directly. ♦

◇ **4.7-3.** Let f be a nonconstant polynomial such that $f(0) = f(1)$. Prove that f has a local minimum or a local maximum point somewhere in the open interval $]0, 1[$.

Sketch. f attains both maximum and minimum. (Why?) If both are at ends, then f is constant. (Why?) ◇

Solution. Polynomials are continuous everywhere, so the function f is certainly continuous on the compact domain $[0, 1]$. By the maximum-minimum theorem, it must attain both its maximum and its minimum on that set. Since $f(0) = f(1)$, the only way that both of these can be at the ends is for the maximum and minimum to be the same. f would have to be constant, but it has been assumed to be nonconstant. At least one of the maximum or minimum must occur in the interior and thus be a local extremum. ♦

◇ **4.7-5.** Let f be continuous on $[3, 5]$ and differentiable on $]3, 5[$, and suppose that $f(3) = 6$ and $f(5) = 10$. Prove that, for some point x_0 in the open interval $]3, 5[$, the tangent line to the graph of f at x_0 passes through the origin. Illustrate your result with a sketch.

Suggestion. Consider the function $f(x)/x$. ◇

Solution. The equation of the line tangent to the graph of f at the point $(x_0, f(x_0)$ is $y = f(x_0) + f'(x_0)(x - x_0)$. For this to pass through the origin we must have $0 = f(x_0) - xf'(x_0)$. A medium sized amount of meditation and a bit of inspiration might remind one that this looks like the numerator of the derivative of the function $g(x) = f(x)/x$ computed by the quotient rule. This function is continuous and differentiable on $[3, 5]$, and $g(3) = g(5) = 2$. So Rolle's Theorem applies and says that there is a point x_0 in the interval at which $g'(x_0) = 0$. But $g'(x) = (xf(x) - f(x))/x^2$. For $g'(x_0) = 0$, we must have $x_0 f'(x_0) = f(x_0)$. This is exactly the condition we needed. See Figure 4-9. ♦

◇ **4.8-7.** Let $f : [0, 1] \to \mathbb{R}$, $f(x) = 1$ if $x = 1/n$, n an integer, and $f(x) = 0$ otherwise.

(a) Prove that f is integrable.

(b) Show that $\int_0^1 f(x)\,dx = 0$.

Sketch. $L(f, P) = 0$ for every partition of $[0, 1]$. (Why?) Now take a partition with the first interval $[x_0, x_1] = [0, \sqrt{2}/n]$ and the others of length no more than $1/n^2$. Show that $U(f, P) \le (\sqrt{2}/n) + (n - 1)/n^2$. ◇

Solution. Let $P = \{0 = x_0 < x_1 < x_2 < \cdots < x_m = 1\}$ be any partition of $[0, 1]$. Since $f(x) = 0$ except at the isolated points $1, 1/2, 1/3, \ldots$, there are points in every subinterval where $f(x) = 0$. Since $f(x)$ is never negative, we have $m_j = \inf\{f(x) \mid x \in [x_{j-1}, x_j]\} = 0$ for each $j = 1, 2, 3, \ldots$. So

$$0 = L(f, P) \le \underline{\int_0^1} f(x)\,dx.$$

Now let n be any integer larger that 3 and $Q = \{x_0 < x_1 < x_2 < \cdots < x_m\}$ be a partition of $[0,1]$ with $x_0 = 0$, $x_1 = \sqrt{2}/n$, $x_m = 1$, and $x_j - x_{j-1} < 1/n^2$ for $j = 2, 3, \ldots, m$. The points $1/n, 1/(n+1), 1/(n+2), \ldots$ are all located in $[x_0, x_1]$. So $M_1 = \sup\{f(x) \mid x \in [x_0, x_1]\} = 1$, and this subinterval contributes an amount $M_1(x_1 - x_0) = \sqrt{2}/n$ to the upper sum. Outside this interval, f is nonzero only at the points $1, 1/2, 1/3, \ldots, 1/(n-1)$. There are $n-1$ of these points, and the subintervals in which they occur each have length no more than $1/n^2$. So the total contribution to the upper integral from these subintervals is no larger than $(n-1)/n^2$. Thus

$$\overline{\int_0^1} f(x)\,dx \le U(f,Q) \le \frac{\sqrt{2}}{n} + \frac{n-1}{n^2}.$$

Since this tends to 0 as n increases, and the inequality holds for every $n > 3$, we conclude that

$$0 = L(f,P) \le \underline{\int_0^1} f(x)\,dx \le \overline{\int_0^1} f(x)\,dx \le U(f,Q) \le 0.$$

The upper and lower integrals must both be 0. Since they are equal, f is integrable on $[0,1]$, and

$$\int_0^1 f(x)\,dx = \underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = 0. \qquad \blacklozenge$$

◇ **4.8-8.** Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $|f(x)| \le M$. Let $F(x) = \int_a^x f(t)\,dt$. Prove that $|F(y) - F(x)| \le M\,|y - x|$. Deduce that F is continuous. Does this check with Example 4.8.10?

Solution. Using Proposition 4.8.5(iv), we have

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right|.$$

If $x < y$, this is $\left|\int_x^y f(t)\,dt\right|$. If $y < x$, then it is $\left|-\int_y^x f(t)\,dt\right|$. In either case, it is $\left|\displaystyle\int_{\min(x,y)}^{\max(x,y)} f(t)\,dt\right|$. From the observation at the top of page 207 of the text following Proposition 4.8.5, we have

$$|F(y) - F(x)| = \left| \int_{\min(x,y)}^{\max(x,y)} f(t)\,dt \right| \le \int_{\min(x,y)}^{\max(x,y)} |f(t)|\,dt.$$

But $|f(t)| \le M$ for all t. With this, 4.8.5(iii), and the result of Exercise 4.8-4, we have

$$\begin{aligned} |F(y) - F(x)| &\le \int_{\min(x,y)}^{\max(x,y)} |f(t)|\,dt \le \int_{\min(x,y)}^{\max(x,y)} M\,dt \\ &\le M(\max(x,y) - \min(x,y)) = M\,|x - y| \end{aligned}$$

as claimed.

This inequality holds for all x and y in $[a,b]$. So F satisfies a Lipschitz condition on $[a,b]$ with constant M. This implies f is uniformly continuous on $[a,b]$. If $M = 0$, then F must be constant and is certainly uniformly continuous. If $M > 0$ and $\varepsilon > 0$, put $\delta = \varepsilon/M$. If x and y are in $[a,b]$ and $|x - y| < \delta$, we have $|F(x) - F(y)| \le M\,|x - y| < M\delta = \varepsilon$. So F is uniformly continuous on $[a,b]$.

In Example 4.8.10 we had

$$f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 < x \le 2 \end{cases} \quad \text{and } F(x) = \begin{cases} 0, & 0 \le x \le 1 \\ x-1, & 1 < x \le 2 \end{cases} .$$

Although f is not continuous on $[0, 2]$, the indefinite integral $F(x)$ is. See Figure 4.8-3 of the text. ♦

◇ **4E-2.** (a) Prove that if $f : A \to \mathbb{R}^m$ is continuous and $B \subset A$, then the restriction $f|B$ is continuous.

(b) Find a function $g : A \to \mathbb{R}$ and a set $B \subset A$ such that $g|B$ is continuous but g is continuous at no point of A.

Solution. (a) Suppose $\varepsilon > 0$ and $x_0 \in B$. Then $x_0 \in A$, so there is a $\delta > 0$ such that $\| f(x) - f(x_0) \| < \varepsilon$ whenever $x \in A$ and $\| x - x_0 \| < \delta$. If $x \in B$, then it is in A, so

$$(x \in B \text{ and } \| x - x_0 \| < \delta) \implies \| f(x) - f(x_0) \| .$$

So f is continuous at x_0. Since x_0 was arbitrary in B, f is continuous on B.

(b) Let $A = \mathbb{R}$, $B = \mathbb{Q}$, and define $g : A \to \mathbb{R}$ by $g(x) = 1$ if $x \in \mathbb{Q}$ and $g(x) = 0$ if $x \notin \mathbb{Q}$. The restriction of g to $B = \mathbb{Q}$ is constantly equal to 1 on B. So it is continuous on B. (See Exercise 4E-1(b).) But, if $x_0 \in \mathbb{R}$, then there are rational and irrational points in every short interval around x_0. So g takes the values 1 and 0 in every such interval. The values of $g(x)$ cannot be forced close to any single value by restricting to a short interval around x_0. So, as a function on $\mathbb{R}$, g is not continuous at x_0. This is true for every $x_0 \in \mathbb{R}$. ♦

◇ **4E-7.** Consider a compact set $B \subset \mathbb{R}^n$ and let $f : B \to \mathbb{R}^m$ be continuous and one-to-one. Then prove that $f^{-1} : f(B) \to B$ is continuous. Show by example that this may fail if B is connected but not compact. (To find a counterexample, it is necessary to take $m > 1$.)

Sketch. Suppose C is a closed subset of B. Then C is compact. (Why?) So $f(C)$ is closed. (Why?) Thus f^{-1} is continuous. (Why?) For a counterexample with $n = 2$ consider $f : [0, 2\pi[\to \mathbb{R}^2$ given by $f(t) = (\sin t, \cos t)$. ◇

Solution. FIRST PROOF: We use the characterization of continuity in terms of closed sets. To show that $f^{-1} : f(B) \to B$ is continuous on $f(B)$, we need to show that if C is a closed subset of the metric space B, then $(f^{-1})^{-1}(C)$ is closed relative to $f(B)$. Since B is a compact subset of $\mathbb{R}^n$, it is closed, and a subset C of it is closed relative to B if and only if it is closed in $\mathbb{R}^n$. Since it is a closed subset of the compact set B it is closed. (In $\mathbb{R}^n$ this follows since it is closed and bounded. However, it is true more generally. See Lemma 2 to the proof of the Bolzano-Weierstrass Theorem, 3.1.3, at the end of Chapter 3: *A closed subset of a compact space is compact.*) Since C is a compact subset of B and f is continuous on B and hence on C, the image $f(C)$ is compact. Since it is a compact subset of a metric space, it is closed. (See Lemma 1 to the proof of 3.1.3.) But since f is one-to-one, $f(C) = (f^{-1})^{-1}(C)$. Thus $(f^{-1})^{-1}(C)$ is closed for every closed subset C of $f(B)$. The inverse f^{-1} is thus a continuous function from $f(B)$ to B.

SECOND PROOF: Here is a proof using sequences. Suppose $y \in f(B)$ and $\langle y_k \rangle_1^\infty$ is a sequence in $f(B)$ with $y_k \to y$. We want to show that $x_k = f^{-1}(y_k) \to x = f^{-1}(y)$ in B. Since B is compact, there is a subsequence $x_{k(1)}, x_{k(2)}, x_{k(3)}, \ldots$ converging to some point $\hat{x} \in B$. Since f is continuous on B, we must have $y_{k(j)} = f(x_{k(j)}) \to f(\hat{x})$. But $y_{k(j)} \to y$. Since limits are unique in the metric space $f(B)$, we must have $y = f(\hat{x})$. But $y = f(x)$ and f is one-to-one, so $x = \hat{x}$. Not only does this argument show that there must be some subsequence of the x_k converging to x, it shows that x is the only possible limit of a subsequence. Since B is compact, every subsequence would have to have a sub-subsequence converging to something, and the only possible "something" is x. Thus $x_k \to x$ as needed.

If the domain is not compact, for example the half-open interval $B = [0, 2\pi[$, then we can get a counterexample. The map $f : [0, 2\pi[\to \mathbb{R}^2$ given by $f(t) = (\sin t, \cos t)$ takes $[0, 2\pi[$ onto the unit circle. The point $(0, 1)$ has preimages near 0 and near 2π. So the inverse function is not continuous at $(0, 1)$.

It turns out that a continuous map from a half-open interval one-to-one into $\mathbb{R}$ must have a continuous inverse. Challenge: Prove it. ♦

◇ **4E-8.** Define maps $s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as addition and scalar multiplication defined by $s(x, y) = x + y$ and $m(\lambda, x) = \lambda x$. Show that these mappings are continuous.

Solution. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then $s(x, y) = x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, and $m(\lambda, x) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$.

We compute

$$\begin{aligned}
\| s(x,y) - s(u,v) \|_{\mathbb{R}^n} &= \| (x+y) - (u+v) \|_{\mathbb{R}^n} = \| (x-u) + (y-v) \|_{\mathbb{R}^n} \\
&\le \| x - u \|_{\mathbb{R}^n} + \| y - v \|_{\mathbb{R}^n} \\
&\le 2\sqrt{\| x - u \|_{\mathbb{R}^n}^2 + \| y - v \|_{\mathbb{R}^n}^2} \\
&\le 2 \| (x-u, y-v) \|_{\mathbb{R}^n \times \mathbb{R}^n} = 2 \| (x,y) - (u,v) \|_{\mathbb{R}^n \times \mathbb{R}^n} .
\end{aligned}$$

So, if $\| (x,y) - (u,v) \|_{\mathbb{R}^n \times \mathbb{R}^n} < \varepsilon/2$, then $\| s(x,y) - s(u,v) \|_{\mathbb{R}^n} < \varepsilon$. So s is continuous.

Fix $(\mu, u) \in \mathbb{R} \times \mathbb{R}^n$. We have

$$\| (\lambda, x) - (\mu, u) \|_{\mathbb{R}^n \times \mathbb{R}^n}^2 = \| (\lambda - \mu, x - u) \|_{\mathbb{R}^n \times \mathbb{R}^n}^2 = (\lambda - \mu)^2 + \| x - u \|_{\mathbb{R}^n}^2 .$$

Thus if $\| (\lambda, x) - (\mu, u) \|_{\mathbb{R}^n \times \mathbb{R}^n} < \delta$ then $|\lambda - \mu| < \delta$ and $\| x - u \|_{\mathbb{R}^n} < \delta$.

$$\begin{aligned}
\| m(\lambda, x) - m(\mu, u) \|_{\mathbb{R}^n} &= \| \lambda x - \mu u \|_{\mathbb{R}^n} = \| \lambda x - \lambda u + \lambda u - \mu u \|_{\mathbb{R}^n} \\
&\le \| \lambda x - \lambda u \|_{\mathbb{R}^n} + \| \lambda u - \mu u \|_{\mathbb{R}^n} = |\lambda| \, \| x - u \|_{\mathbb{R}^n} + |\lambda - \mu| \, \| u \|_{\mathbb{R}^n} \\
&\le |\lambda| \delta + \delta \| u \|_{\mathbb{R}^n} \le (|\mu| + \delta)\delta + + \delta \| u \|_{\mathbb{R}^n} .
\end{aligned}$$

If we require that $\delta < 1$ and $\delta < \varepsilon/2(|\mu| + 1)$ and $\delta < \varepsilon/2(\| u \| + 1)$, we have

$$\begin{aligned}
\| m(\lambda, x) - m(\mu, u) \|_{\mathbb{R}^n} &\le (|\mu| + \delta)\delta + \delta \| u \|_{\mathbb{R}^n} \\
&\le (|\mu| + 1)\frac{\varepsilon}{2(|\mu| + 1)} + \frac{\varepsilon}{2(\| u \| + 1)} \| u \| < \varepsilon.
\end{aligned}$$

Thus m is continuous. ♦

◇ **4E-9.** Prove the following "gluing lemma": *Let $f : [a,b[\to \mathbb{R}^m$ and $g : [b,c] \to \mathbb{R}^m$ be continuous. Define $h : [a,c] \to \mathbb{R}^m$ by $h = f$ on $[a,b]$ and $h = g$ on $[b,c]$. If $f(b) = g(b)$, then h is continuous.* Generalize this result to sets A, B in a metric space.

Suggestion. Show that If F is closed in $\mathbb{R}^m$, then $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$ and so is closed. ◊

Solution. Put $A = [a,b]$ and $B = [b,c]$. Let $y_0 = f(b) = g(b)$. Then the function h is well-defined since it makes no difference whether we use f or g to define h at the point b in $A \cap B$. and suppose F is a closed set in $\mathbb{R}^m$. Then

$$\begin{aligned} h^{-1}(F) &= h^{-1}(F) \cap (A \cup B) = (h^{-1}(F) \cap A) \cup (h^{-1}(F) \cap B) \\ &= (f^{-1}(F) \cap A) \cup (g^{-1}(F) \cap B) \end{aligned}$$

Since f and g are continuous and F, $[a,b]$, and $[b,c]$ are closed, this set is closed. The inverse image of every closed set is closed, so h is continuous.

For the generalization, suppose A and B are closed sets in $\mathbb{R}^n$ and that $f : A \to \mathbb{R}^m$ and $g : B \to \mathbb{R}^m$ are continuous and that $f(x) = g(x)$ for $x \in A \cap B$. Define h on $A \cup B$ by putting $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$. After observing that h is well-defined since f and g agree on the intersection, the proof of continuity is the same.

Note that some sort of assumption needs to be made about the sets A and B. Otherwise we could take something like $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. If we put $f(x) = 1$ for all x in A and $g(x) = 0$ for all x in B, then, since $A \cap B = \emptyset$, h is still well-defined, but h is not continuous at any point of $\mathbb{R} = A \cup B$. ♦

◇ **4E-12.** (a) A map $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called *Lipschitz on A* if there is a constant $L \geq 0$ such that $\| f(x) - f(y) \| \leq L \| x - y \|$, for all $x, y \in A$. Show that a Lipschitz map is uniformly continuous.

Solution. (a) Suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and there is a constant $L \geq 0$ such that $\| f(x) - f(y) \| \leq L \| x - y \|$ for all x, $y \in A$. If $L = 0$, then $\| f(x) - f(y) \| = 0$ for all x and y in A. f must be a constant function on A and so is certainly uniformly continuous. If $L > 0$ and $\varepsilon > 0$, put $\delta = \varepsilon / L$. If x and y are in A and $\| x - y \| < \delta$ we have

$$\| f(x) - f(y) \| \leq L \| x - y \| < L\delta = \varepsilon.$$

So f is uniformly continuous on A.

◇ **4E-14.** (a) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x \to 0} \lim_{y \to 0} f(x,y) \quad \text{and} \quad \lim_{y \to 0} \lim_{x \to 0} f(x,y)$$

exist but are not equal.

(b) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits in (a) exist and are equal but f is not continuous. [Hint: $f(x,y) = xy/(x^2+y^2)$ with $f = 0$ at $(0,0)$.]

(c) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous. [Hint: Consider the function given in polar coordinates by $r \tan(\theta/4)$, $0 \leq r < \infty$, $0 \leq \theta < 2\pi$.]

Solution. (a) Let $f(0,0) = 0$, and for other points, put $f(x,y) = x^2/(x^2 + y^2)$. For fixed nonzero x we have $\lim_{y\to 0} f(x,y) = x^2/x^2 = 1$. So

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 1}(0) = 1.$$

For fixed, nonzero y, we have $\lim_{x\to 0} f(x,y) = 0/y^2 = 0$. So

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}(0) = 0.$$

(b) Let $f(0,0) = 0$, and for other points, put $f(x,y) = xy/(x^2 + y^2)$. For fixed nonzero x we have $\lim_{y\to 0} f(x,y) = 0/x^2 = 0$. So

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}(0) = 0.$$

For fixed, nonzero y, we have $\lim_{x\to 0} f(x,y) = 0/y^2 = 0$. So

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}(0) = 0.$$

But, if we look at the values of f along the line $y = x$, we find $f(x,x) = x^2(x^2 + x^2) = 1/2$. Since there are such points in every neighborhood of the origin, and $f(0,0) = 0$, f is not continuous at the origin.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given in polar coordinates by $r\tan(\vartheta/4)$, $0 \le r < \infty$ and $0 \le \vartheta < 2\pi$. Any line through the origin can be parameterized as $\gamma(t) = (t\cos\vartheta_0, t\sin\vartheta_0)$ for $-\infty < t < \infty$ and fixed ϑ_0 with $-\pi/2 < \theta_0 \le \pi/2$. Along such a line, the values of f are given by $t\tan(\vartheta_0/4)$ which is a continuous function of t. So f is continuous along every line

◇ **4E-15.** Let $f_1, \ldots, f_N$ be functions from $A \subset \mathbb{R}^n$ to $\mathbb{R}$. Let m_i be the maximum of f_i, that is, $m_i = \sup(f_i(A))$. Let $f = \sum f_i$ and $m = \sup(f(A))$. Show that $m \le \sum m_i$. Give an example where equality fails.

Suggestion. For an example with inequality, try $f_1(x) = x$ and $f_2(x) = 1 - x$ on $[0,1]$. ◇

Solution. If $x \in A$, then $f_k(x) \le \sup\{f_k(x) \mid x \in A\} = m_k$ for each $k = 1, 2, \ldots N$. So

$$\begin{aligned} f(x) &= f_1(x) + f_2(x) + \cdots + f_N(x) \\ &\le m_1 + f_2(x) + \cdots + f_N(x) \\ &\le m_1 + m_2 + \cdots + f_N(x) \\ &\vdots \\ &\le m_1 + m_2 + \cdots + m_N. \end{aligned}$$

This holds for every $x \in A$, so $m = \sup\{f(x) \mid x \in A\} \le m_1 + m_2 + \cdots + m_N$ as claimed.

The inequality can be strict. Let $A = [0,1] \subseteq \mathbb{R}$, and set $f_1(x) = x$ and $f_2(x) = 1 - x$. Then $m_1 = \sup\{x \mid x \in [0,1]\} = 1$ and $m_2 = \sup\{1 - x \mid x \in [0,1]\} = 1$. So $m_1 + m_2 = 2$. But $f(x) = f_1(x) + f_2(x) = x + (1 - x) = 1$ for all x. So $m = \sup\{f(x) \mid x \in [0,1]\} = 1$. This is strictly less than 2. ♦

through the origin with the value 0 at the origin. However, if you fix $r_0 > 0$ and move counterclockwise around the circle of radius r_0, the values of f start at 0 on the horizontal axis at $(r_0, 0)$. As you move around the circle, the values are $r_0\tan(\vartheta/4$. As you get close to $(r_0, 0)$ from below the horizontal axis. $\vartheta/4 \to \pi/2$, so the values of f tend to $+\infty$. Thus f is not continuous across the positive horizontal axis. ♦

◇ **4E-18.** Let $A \subset M$ be connected and let $f : A \to \mathbb{R}$ be continuous with $f(x) \neq 0$ for all $x \in A$. Show that $f(x) > 0$ for all $x \in A$ or else $f(x) < 0$ for all $x \in A$.

Solution. If there were a points x_1 and x_2 with $f(x_1) < 0$ and $f(x_2) > 0$, then by the intermediate value theorem, 4.5.1, there would be a point z in A with $f(z) = 0$. By hypothesis, this does not happen. So $f(x)$ must have the same sign for all x in A. ♦

◇ **4E-19.** Find a continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$ and a closed set $A \subset \mathbb{R}^n$ such that $f(A)$ is not closed. In fact, do this when $f : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the x–axis.

Sketch. $A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$; $f(A) = \{x \in \mathbb{R} \mid x \neq 0\}$. ◊

Solution. For $(x, y) \in \mathbb{R}^2$, let $f(x, y) = \pi_1(x, y) = x$. Then f is a continuous map from $\mathbb{R}^2$ to $\mathbb{R}$ (Exercise 4.1-1(b)). Let $A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. This hyperbola is a closed set in $\mathbb{R}^2$, but $f(A) = \mathbb{R}\setminus\{0\}$ which is not closed.

For a second example, let $A \subseteq \mathbb{R}^2$ be the graph of $y = \tan x$ for $-\pi/2 < x < \pi/2$. The curve A is a closed set in $\mathbb{R}^2$, but $f(A)$ is the open interval $]-\pi/2, \pi/2[$ which is not closed. ♦

◇ **4E-21.** Which of the following functions on $\mathbb{R}$ are uniformly continuous?

(a) $f(x) = 1/(x^2 + 1)$.
(b) $f(x) = \cos^3 x$.
(c) $f(x) = x^2/(x^2 + 2)$.
(d) $f(x) = x \sin x$.

Answer. (a) Yes.
(b) Yes.
(c) Yes.
(d) No. ◊

Solution. Parts (a), (b), and (c) can all be handled using Example 4.6.4. Each is differentiable everywhere on $\mathbb{R}$.

(a) If $f(x) = 1/(x^2 + 1)$, then $f'(x) = -2x/(x^2 + 1)^2$. The denominator is always at least as large as 1. So, if $|x| \le 1$, then $|f'(x)| \le 2$. If $|x| > 1$, then $|f'(x)| \le \left|2x/x^4\right| = 2/\left|x^3\right| < 2$. So $|f'(x)| \le 2$ for all $x \in \mathbb{R}$, and f is uniformly continuous on $\mathbb{R}$ by Example 4.6.4.
(b) If $f(x) = \cos^3 x$, then $|f'(x)| = \left|-3\cos^2 x \sin x\right| \le 3$ for all real x since $|\sin x| \le 1$ and $|\cos x| \le 1$ for all real x. Again, f is uniformly continuous on $\mathbb{R}$ by Example 4.6.4.
(c) If $f(x) = x^2/(x^2 + 2)$, then

$$|f'(x)| = \left|\frac{4x}{x^4 + 4x^2 + 4}\right| \le \left|\frac{x}{x^2 + 1}\right|.$$

The denominator is always at least as large as 1, so, if $|x| \le 1$, then $|f'(x)| \le 1$. If $x > 1$, then $|f'(x)| \le \left|x/x^2\right| = 1/|x| < 1$. So $|f'(x)| \le 1$ for all $x \in \mathbb{R}$, and f is uniformly continuous on $\mathbb{R}$ by Example 4.6.4.

(d) The function $f(x) = x \sin x$ is continuous on $\mathbb{R}$ but not absolutely continuous. Let $\varepsilon = 1$. We will show that no $\delta > 0$ can satisfy the definition of uniform continuity everywhere in $\mathbb{R}$. Let a be any number with $0 < a < \min(\delta, \pi/4)$. Then $\sin a > 0$. Pick an integer k large enough so that $2k\pi \sin a > 1$. Let $x = 2k\pi$ and $y = 2k\pi + a$. Then $|x - y| = a < \delta$, but

$$\begin{aligned}|f(y) - f(x)| &= |(2k\pi + a)\sin(2k\pi + a) - \sin(2k\pi)| \\ &= (2k\pi + a)\sin(a) \\ &> 1.\end{aligned}$$

No single $\delta > 0$ can work for $\varepsilon = 1$ in the definition of uniform continuity everywhere in $\mathbb{R}$. So f is not uniformly continuous on $\mathbb{R}$. ♦

◇ **4E-23.** Let X be a compact metric space and $f : X \to X$ an isometry; that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is a bijection.

Sketch. To show "onto" suppose $y_1 \in \mathcal{X} \setminus f(\mathcal{X})$ and consider the sequence $y_2 = f(y_1)$, $y_3 = f(y_2)$, ... ◇

Solution. If $f(x) = f(y)$, then $0 = d(f(x), f(y)) = d(x, y)$, so $x = y$. Thus f is one-to-one. If $\varepsilon > 0$, let $\delta = \varepsilon$. If $d(x, y) < \delta$, then $d(f(x), f(y)) = d(x, y) < \delta = \varepsilon$, so f is continuous, in fact uniformly continuous, on $\mathcal{X}$. It remains to show that f maps $\mathcal{X}$ onto $\mathcal{X}$. Since $\mathcal{X}$ is compact and f is continuous, the image, $f(\mathcal{X})$ is a compact subset of the metric space $\mathcal{X}$. So it must be closed. Its complement, $\mathcal{X} \setminus f(\mathcal{X})$ must be open. If there were a point x in $\mathcal{X} \setminus f(\mathcal{X})$, then there would be a radius $r > 0$ such that $D(x, r) \subseteq \mathcal{X} \setminus f(\mathcal{X})$. That is, $y \in f(\mathcal{X})$ implies $d(y, x) > r$. Consider the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$. For positive integer k, let f^k denote the composition of f with itself k times. If n and p are positive integers, then

$$d(x_{n+p}, x_n) = d(f^n \circ f^p(x), f^n(x)) = d(f^p(x), x) > r$$

since $f^p(x) \in f(\mathcal{X})$. The points in the sequence are pairwise separated by distances of at least r. This would prevent any subsequence from converging. But $\mathcal{X}$ is sequentially compact by the Bolzano-Weierstrass Theorem. So there should be a convergent subsequence. This contradiction shows that there can be no such starting point x for our proposed sequence. The complement $\mathcal{X} \setminus f(\mathcal{X})$ must be empty. So $f(\mathcal{X}) = \mathcal{X}$ and f maps $\mathcal{X}$ onto $\mathcal{X}$ as claimed. ♦

◇ **4E-28.** Let $f :]0, 1[\to \mathbb{R}$ be uniformly continuous. Must f be bounded?

Answer. Yes. ◇

Solution. If f were not bounded on $]0, 1[$, we could inductively select a sequence of points $\langle x_k \rangle_1^\infty$ in $]0, 1[$ such that $|f(x_{k+1})| > |f(x_k)| + 1$ for each k. In particular, we would have $|f(x_k) - f(x_j)| > 1$ whenever $k \neq j$. But the points x_k are all in the compact interval $[0, 1]$, so there should be a subsequence converging to some point in $[0, 1]$. This subsequence would have to be a Cauchy sequence, so no matter how small a positive number δ were specified, we could get points x_k and x_j in the subsequence with $|x_k - x_j| < \delta$ and $|f(x_k) - f(x_j)| > 1$. This contradicts the uniform continuity of f on $]0, 1[$. So the image $f(]0, 1[)$ must, in fact, be bounded.

If we knew the result of Exercise 4E-24(c), then we would know that f has a unique continuous extension to the closure, $[0, 1]$. There is a continuous $g : [0, 1] \to \mathbb{R}$ such that $g(x) = f(x)$ for all x in $]0, 1[$. Since g is continuous on the compact domain $[0, 1]$, the image $g([0, 1])$ is compact and hence bounded. So $f(]0, 1[) = g(]0, 1[) \subseteq g([0, 1])$ is bounded. ♦

◇ **4E-30.** (a) Let $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x}$. Prove that f is uniformly continuous.

(b) Let $k > 0$ and $f(x) = (x - x^k)/\log x$ for $0 < x < 1$ and $f(0) = 0$, $f(1) = 1 - k$. Show that $f : [0, 1] \to \mathbb{R}$ is continuous. Is f uniformly continuous?

Suggestion. (a) Use Theorem 4.6.2 to show that f is uniformly continuous on $[0, 3]$ and Example 4.6.4 to show that it is uniformly continuous on $[1, \infty[$. Then combine these results.

(b) Use L'Hôpital's Rule. ◇

Solution. (a) Let $\varepsilon > 0$. We know that $f(x) = \sqrt{x}$ is continuous on $[0, \infty[$, so it is certainly continuous on the compact domain $[0, 3]$. By the uniform continuity theorem, 4.6.2, it is uniformly continuous on that set. There is a $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in $[0, 3]$ and $|x - y| < \delta_1$.
We also know that f is differentiable for $x > 0$ with $f'(x) = 1/(2\sqrt{x})$. So $|f'(x)| \leq 1/2$ for $x \geq 1$. As in Example 4.6.4, we can use the mean value theorem to conclude that if $\delta_2 = 2\varepsilon$, and x and y are in $[1, \infty[$ with $|x - y| < \delta_2$, then there is a point c between x and y such that $|f(x) - f(y)| = |f'(c)(x - y)| < (1/2)(2\varepsilon) = \varepsilon$.
Now take advantage of the overlap of our two domains. If x and y are in $[0, \infty[$ and $|x - y| < \delta = \min(1, \delta_1, \delta_2)$, then either x and y are both in $[0, 3]$ or both are in $[1, \infty[$ or both. If they are both in $[0, 3]$, then $|f(x) - f(y)| < \varepsilon$ since $|x - y| < \delta_1$. If they are both in $[1, \infty[$, then $|f(x) - f(y)| < \varepsilon$ since $|x - y| < \delta_2$. In either case, $|f(x) - f(y)| < \varepsilon$. So f is uniformly continuous on $[0, \infty[$ as claimed.

(b) Suppose k is a positive integer and $f(x) = (x - x^k)/\log x$ for $0 < x < 1$, $f(0) = 0$, and $f(1) = 1 - k$. The numerator, $x - x^k$, is continuous for all x. The denominator, $\log x$, is continuous for $x > 0$. So f is continuous on $x > 0$ except possibly at $x = 1$ where the denominator is 0, However, the numerator is also 0 at $x = 1$. To apply L'Hôpital's Rule, we consider the ratio of the derivatives

$$\frac{1 - kx^{k-1}}{1/x} = x - kx^k \to 1 - k = f(1) \quad \text{as} \quad x \to 1.$$

By L'Hôpital's Rule, $\lim_{x\to 1}(x - x^k)/\log x = \lim_{x\to 1} f(x)$ also exists and is equal to $f(1)$. So f is continuous at 1. As $x \to 0^+$, the numerator of $f(x)$ tends to 0 and the denominator to $-\infty$. So $\lim_{x\to 0^+} f(x) = 0 = f(0)$. So f is continuous from the right at 0. So f is continuous on $[0, \infty[$ and on the smaller domain $[0, 1]$. Since the latter is compact, f is uniformly continuous on it by the uniform continuity theorem, 4.6.2. ◆

◇ **4E-34.** Assuming that the temperature on the surface of the earth is a continuous function, prove that on any great circle of the earth there are two antipodal points with the same temperature.

Solution. View the great circle as a circle of radius R in the xy-plane. If t is a real number, then $(R\cos t, R\sin t)$ and $(R\cos(t + \pi), R\sin(t + \pi))$ are antipodal points (at opposite ends of a diameter). Let $f(t)$ be the temperature at $(R\cos t, R\sin t)$, and $g(t) = f(t) - f(t + \pi)$. We want to show that there is a t_0 with $g(t_0) = 0$. But $g(t+\pi) = f(t+\pi) - f(t+2\pi) = f(t + \pi) - f(t) = -g(t)$. If $g(t)$ is not zero, then $g(t)$ and $g(t + \pi)$ have opposite sign. Since f is continuous, so is g, and the intermediate value theorem guarantees a point t_0 at which $g(t_0) = 0$ just as we need. ◆

◇ **4E-36.** Show that $\{(x, \sin(1/x)) \mid x > 0\} \cup (\{0\} \times [-1, 1])$ in $\mathbb{R}^2$ is connected but not path-connected.

Solution. Let $A = \{(x, \sin(1/x)) \mid x > 0\}$ and $B = \{0\} \times [-1, 1] = \{(0, y) \mid -1 \le y \le 1\}$. Let $C = A \cup B$. We are asked to show that C is connected but not path-connected.

Each of A and B are path-connected, and so connected, subsets of $\mathbb{R}^2$. Suppose U and V were open sets with C contained in their union and $U \cap V \cap C = \emptyset$. If $A \cap U$ and $A \cap V$ were both nonempty, then U and V would disconnect A. But A is connected. So one of these must be empty. Similarly, one of $B \cap U$ and $B \cap V$ must be empty. Say $B \cap V = \emptyset$ and $B \subseteq U$. Since U is open and $(0, 0) \in U$, the point $(1/2\pi n, 0)$ is also in U for large enough integer n. But these points are in A. So $A \cap U$ is not empty. So $A \cap V$ is empty, and $A \subseteq U$. So $C = A \cup B \subseteq U$. The sets U and V cannot disconnect C. So C must be connected.

Suppose $\gamma : [0, 1] \to C$ were a continuous path with $\gamma(0) = (1/2\pi, 0)$ and $\gamma(1) = (0, 0)$. Since γ is continuous on the compact domain $[0, 1]$, it would be uniformly continuous, and there would be a $\delta > 0$ such that $0 \le s \le t < \delta$ implies $\| \gamma(s) - \gamma(t) \| < 1/2$. But this gets us into trouble. As the path moves from $(1/2\pi, 0)$ to $(0, 0)$, it must pass through the points $v_n = (1/2\pi n, 0)$ and $w_n = (1/(2\pi n + (\pi/2)), 1)$. We could inductively select preimages $0 = s_1 < t_1 < s_2 < t_2 < s_3 < \cdots \to 1$ such that $\gamma(s_n) = v_n$ and $\gamma(t_n) = w_n$. For large enough n, both s_n and t_n are within δ of 1, so their images should be separated by less than 1/2. But $\| v_n - w_n \| \ge 1$. So there can be no such path. ♦

◇ **4E-44.** Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable and suppose for every a, b with $0 \le a < b \le 1$ there is a c, $a < c < b$, with $f(c) = 0$. Prove $\int_0^1 f = 0$. Must f be zero? What if f is continuous?

Suggestion. Show that the upper and lower sums are both 0 for every partition of $[0, 1]$. Consider a function which is 0 except at finitely many points. ◇

Solution. Since f is integrable on $[0, 1]$, the upper and lower integrals are the same and are equal to the integral. Let $P = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ be any partition of $[0, 1]$. For each subinterval $[x_{j-1}, x_j]$ there is a point c_j in it with $f(c_j) = 0$. So

$$m_j = \inf\{f(x) \mid x \in [x_{j-1}, x_j]\} \le 0 \le \sup\{f(x) \mid x \in [x_{j-1}, x_j]\} = M_j.$$

So

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) \le 0 \le \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = U(f, P).$$

This is true for every partition of $[0, 1]$. So

$$\begin{aligned} \int_0^1 f(x)\,dx = \underline{\int_0^1} f(x)\,dx &= \sup_{P \text{ a partition of } [0,1]} L(f, P) \le 0 \\ &\le \inf_{P \text{ a partition of } [0,1]} U(f, P) = \overline{\int_0^1} f(x)\,dx = \int_0^1 f(x)\,dx. \end{aligned}$$

So we must have $\int_0^1 f(x)\,dx = 0$.

The function f need not be identically 0. We could, for example, have $f(x) = 0$ for all but finitely many points at which $f(x) = 1$.

If f is continuous and satisfies the stated condition, then f must be identically 0. Let $x \in [0, 1]$. By hypothesis there is, for each integer $n > 0$, at least one point c_n in $[0, 1]$ with $x-(1/n) \leq c_n \leq x+(1/n)$ and $f(c_n) = 0$. Since $c_n \to 0$ and f is continuous, we must have $0 = f(c_n) \to f(x)$. So $f(x) = 0$. ♦