Exercises within a Section: 1.1-2 means Section 1.1 problem #2 etc...

☞**Watch out! To save paper and spaces, some solutions may not be in the proper order. You should be able to find them.**（解答裡題目的順序可能不會照書上的順序，請大家在「該出現的地方」找不到時往後翻一下）

☞**You are required to reproduce or to paraphrase of the "Solution" (NOT the "Sketch") to a problem.** （考試寫證明的時候，是要寫 Solution 的部份，而不是 Sketch 的部份！Sketch 的部份是告訴大家證明的 idea 是什麼。）

☞**Do not skip over problems that you think are complicated. We can still ask you part of the steps in a test.**（不要跳過你覺得證明太複雜的問題，我們考試時仍可能會考你！）

◇ **2.1-4.** Let $B \subset \mathbb{R}^n$ be any set. Define $C = \{x \in \mathbb{R}^n \mid d(x, y) < 1$ for some $y \in B\}$. Show that C is open.

Suggestion. The set C is a union of open balls. ◇

Solution. Let $A = \bigcup_{y \in B} D(y, 1)$.Since each of the disks $D(y, 1)$ is an open subset of $\mathbb{R}^n$ and A is their union, we know that A is an open set by Proposition 2.1.3(ii). We will show that C is open by showing that $C = A$. If $y \in B$ and x is any point in $\mathbb{R}^n$ with $\| x - y \| < 1$, then $x \in C$ by the definition of the set C. So $D(y, 1) \subseteq C$ for every y in B. Thus $A = \bigcup_{y \in B} D(y, 1) \subseteq C$. In the other direction, if $x \in C$, then there is a point y in B with $d(x, y) = \| x - y \| < 1$. So $x \in D(y, 1) \subseteq A$. Thus $C \subseteq A$. We have containment in both directions, so $C = A$. Thus C is an open set as claimed. ♦

◇ **2.1-6.** Show that $\mathbb{R}^2$ with the taxicab metric has the same open sets as it does with the standard metric.

Sketch. The key idea is that each "taxicab disk" centered at a point P contains a "Euclidean disk" centered at P and also the reverse. ◇

Solution. Recall that the "taxicab metric" on $\mathbb{R}^2$ is defined for points $P = (a, b)$ and $Q = (x, y)$ by $d_1(Q, P) = |x - a| + |y - b|$. The Euclidean metric is defined by $d_2(Q, P) = \sqrt{(x - a)^2 + (y - b)^2}$. Euclidean disks look like the interior of a usual Euclidean circle, while a "taxicab disk" looks like the inside of a "diamond". The key idea is that each taxicab disk centered at P contains a Euclidean disk centered at P and each Euclidean disk centered at P contains a taxicab disk centered at P. See the figure.

$$\begin{aligned} d_1(Q,P)^2 &= (|x-a|+|y-b|)^2 = (x-a)^2 + 2\,|x-a|\cdot|y-b| + (y-b)^2 \\ &\geq (x-a)^2 + (y-b)^2 = d_2(Q,P)^2 \end{aligned}$$

On the other hand, we know that for any real s and t that $2st \leq s^2 + t^2$ since $0 \leq (s-t)^2 = s^2 - 2st + t^2$. With $s = |x-a|$ and $t = |y-b|$, this gives

$$\begin{aligned} d_1(Q,P)^2 &= (|x-a|+|y-b|)^2 = (x-a)^2 + 2\,|x-a|\cdot|y-b| + (y-b)^2 \\ &\leq 2((x-a)^2 + (y-b)^2) = 2\,d_2(Q,P)^2 \end{aligned}$$

Taking square roots gives

$$d_2(Q,P) \leq d_1(Q,P) \leq \sqrt{2}\,d(Q,P) \qquad (*)$$

for every pair of points P and Q in $\mathbb{R}^2$. Denote the Euclidean and taxicab disks around P of radius ρ by

$$\begin{aligned} D_2(P,\rho) &= \{P \in \mathbb{R}^2 \mid d_2(P,Q) < \rho\} \\ D_1(P,\rho) &= \{P \in \mathbb{R}^2 \mid d_1(P,Q) < \rho\}. \end{aligned}$$

If $d_1(P,Q) < \rho$, then by the first part of $(*)$, we also have $d_2(P,Q) < \rho$. So $D_1(P,\rho) \subseteq D_2(P,\rho)$. If $d_2(P,Q) < r$, then by the second part of $(*)$, we also have $d_1(P,Q) < \sqrt{2}\,r$. So $D_2(P,r) \subseteq D_1(P,\sqrt{2}\,r)$. Equivalently $D_2(P,\rho/\sqrt{2}) \subseteq D_1(P,\rho)$ for each $\rho > 0$.

Now Let $S \subseteq \mathbb{R}^2$ and suppose that S is open with respect to the usual Euclidean distance. Let $P \in S$. Then there is an $\rho > 0$ such that $D_2(P,\rho) \subseteq S$. But then $P \in D_1(P,\rho) \subseteq D_2(P,\rho) \subseteq S$. So S contains a taxicab disk around each of its points and must be "taxicab open". On the other hand, if we assume that S is open with respect to the taxicab distance and $P \in S$, then there is an $r > 0$ such that $P \in D_1(P,r) \subseteq S$. The argument above then shows that $P \in D_2(P,r/\sqrt{2}) \subseteq D_1(P,r) \subseteq S$. So S contains a Euclidean disk around each of its points and is open in the usual sense. We have shown

$$S \text{ "Euclidean open"} \implies S \text{ "taxicab open"},$$

and that

$$S \text{ "taxicab open"} \implies S \text{ "Euclidean open"}.$$

So the "open sets" are the same no matter which of the two metrics we choose to use for measuring distance between points in the plane. ♦

◇ **2.2-4.** Is it true that $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$?

Answer. Yes. ◊

Solution. First suppose $x \in \text{int}\,(A \cap B)$. Then there is an $r > 0$ such that $x \in D(x,r) \subseteq A \cap B$. Since $A \cap B \subseteq A$, we have $x \in D(x,r) \subseteq A$. So $x \in \text{int}\,(A)$. Also $A \cap B \subseteq B$, so $x \in D(x,r) \subseteq B$ and $x \in \text{int}\,(B)$. Thus $x \in \text{int}\,(A) \cap \text{int}\,(B)$. Since x was an arbitrary point in $\text{int}\,(A \cap B)$, this shows that $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$. (Notice that we could also employ the result of Exercise 2.2-3 to obtain this inclusion.)

Now suppose that $y \in \text{int}(A) \cap \text{int}(B)$. Then $y \in \text{int}(A)$, so there is an $r_1 > 0$ such that $x \in D(x, r_1) \subseteq A$. Also $y \in \text{int}(B)$. So there is an $r_2 > 0$ such that $y \in D(y, r_2) \subseteq B$. Let r be the smaller of r_1 and r_2 so that $r \le r_1$ and $r \le r_2$. Then $D(y, r) \subseteq D(y, r_1) \subseteq A$, and $D(x, r) \subseteq D(y, r_2) \subseteq B$. So $y \in D(y, r) \subseteq A \cap B$. Thus $y \in \text{int}(A \cap B)$. Since y was an arbitrary point in $\text{int}(A) \cap \text{int}(B)$, this shows that $\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B)$.

We have inclusion in both directions, so $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$ as claimed. ♦

◇ **2.2-5.** Let (M, d) be a metric space, and let $x_0 \in M$ and $r > 0$. Show that

$$D(x_0, r) \subset \text{int}\{y \in M \mid d(y, x_0) \le r\}.$$

Suggestion. Show more generally that if U is open and $U \subseteq A$, then $U \subseteq \text{int}(A)$. (See Exercise 2E-3 at the end of the chapter.) ◇

Solution. Let $A = \{y \in M \mid d(y, x_0) \le r\}$ and $U = D(x_0, r) = \{y \in M \mid d(y, x_0) < r\}$. Then we certainly have $U \subseteq A$, and we know that U is open from Proposition 2.1.2. So the desired conclusion follows immediately from this observation.

Proposition. *If U is an open set and $U \subseteq A$, then $U \subseteq \text{int}(A)$.*

Proof: If x in U, then U is open and $x \in U \subseteq A$. So x satisfies the definition of an interior point of A and $x \in \text{int}(A)$. Since x was an arbitrary point in U, this shows that $U \subseteq \text{int}(A)$ as claimed. ♦

◇ **2.7-3.** Let $A \subset \mathbb{R}^m$, $x_n \in A$, and $x_n \to x$. Show that $x \in \text{cl}(A)$.

Sketch. Use Proposition 2.7.6(ii). ◇

Solution. This is exactly one direction of the equivalence in Proposition 2.7.6(ii) with $M = \mathbb{R}^m$. ♦

◇ **2.8-2.** Let (M, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that M is complete.

Sketch. Use parts (ii) and (iii) of Proposition 2.8.4. ◇

Solution. To show that M is complete we need to show that every Cauchy sequence in M converges to a limit in M. So let $\langle x_n \rangle_1^\infty$ be a Cauchy sequence in M. By Proposition 2.8.4(ii), the sequence is bounded. By hypothesis, this means that it must have a subsequence converging to some point $x \in M$. But then by 2.8.4(iii), the whole sequence converges to x. Thus every Cauchy sequence in M converges to a point in M, so M is complete. ♦

◇ **2.3-5.** Let $S = \{x \in \mathbb{R} \mid x \text{ is irrational}\}$. Is S closed?

Answer. No. ◇

Solution. The point 0 is in the complement of S, but is $r > 0$, then the open interval $]-r, r[$ around 0 must contain irrational points such as $\sqrt{2}/n$ for large integer n. Thus $\mathbb{R} \setminus S = \mathbb{Q}$ is not open. So S is not closed.

More generally, it is not too difficult to show using the irrationality of $\sqrt{2}$ and the Archimedean Principle, that both the rationals and irrationals are scattered densely along the real line in the following sense.

Proposition. *If a and b are real numbers with $a < b$, then there are a rational number r and an irrational number z such that*

$$a < r < b \qquad \text{and} \qquad a < z < b.$$

Proof: The first is accomplished by noting that $b - a > 0$. By the Archimedean principle there is an integer n such that $0 < 1/n < b - a$. By another version of the Archimedean principle there are positive integers k such that $a < k/n$. By the well-ordering of the positive integers, we may assume that k is the smallest such. Then

$$a < \frac{k}{n} = \frac{k-1}{n} + \frac{1}{n} \leq a + \frac{1}{n} < a + (b - a) = b.$$

So $r = k/n$ is a rational number meeting our requirements.

Now use the argument just given twice to obtain a pair of rational numbers r and s with $a < r < s < b$. Since $\sqrt{2} > 1$, we have

$$a < r < r + \frac{s - r}{\sqrt{2}} < r + (s - r) = s$$

The number $z = r + (s - r)/\sqrt{2}$ cannot be rational, for if it were, then $\sqrt{2} = (s - r)/(z - r)$ would also be rational. But it is not. The number z is an irrational number which meets our requirements. ♦

◇ **2.4-6.** Let M be a set with the discrete metric and $A \subset M$ be any subset. Find the set of accumulation points of A.

Answer. $A' = \emptyset$. ◇

Solution. If $x_0 \in M$ then the ball of radius $1/2$ around x_0 contains no points other than x_0 since all distances between unequal points are 1. $D(x_0, 1/2) = \{x_0\}$. Nonetheless, this "ball" is an "open set". Thus no set in M can have any accumulation points. ♦

◇ **2.3-6.** Give an alternative solution of Example 2.3.6 by showing that B is a union of finitely many closed sets.

Solution. Example 2.3.6 asserts that if d is a metric on a set M, and A is a finite subset of M, then the set $B = \{x \in M \mid d(x, y) \leq 1 \text{ for some } y \in A\}$ is closed. For each fixed y in M, let $C_y = \{x \in M \mid d(x, y) \leq 1\}$. Then $B = \bigcup_{y \in A} C_y$. Since A is a finite set, this expresses B as the union of a finite number of sets of type C_y. If we knew that all of these were closed we would have B closed since it would be a finite union of closed sets.

We proceed to show that for each fixed y the set C_y is closed. The argument is essentially the same as that given for Example 2.3.4 as illustrated in Figure 2.3-3 of the text. We simply need to write it down for a general metric space instead of $\mathbb{R}^2$.

Suppose $z \in M \setminus C_y$. then $d(z, y) > 1$. Set $r = d(z, y) - 1 > 0$. If $w \in M$ and $d(w, z) < r$, then

$$d(z, y) \leq d(z, w) + d(w, y) < r + d(w, y) = d(z, y) - 1 + d(w, y),$$

so $1 < d(w, y)$ and $w \in M \setminus C_y$. Thus $D(z, r) \subseteq M \setminus C_y$. The set $M \setminus C_y$ contains a disk around each of its points and so is open. Thus C_y is closed as we needed. ♦

◇ **2.4-3.** Find the accumulation points of the following sets in $\mathbb{R}^2$:

(a) $\{(m, n) \mid m, n \text{ integers}\}$
(b) $\{(p, q) \mid p, q \text{ rational}\}$
(c) $\{(m/n, 1/n) \mid m, n \text{ integers}, n \neq 0\}$
(d) $\{(1/n + 1/m, 0) \mid n, m \text{ integers}, n \neq 0, m \neq 0\}$

Answer. (a) $A' = \emptyset$.
(b) $B' = \mathbb{R}^2$.
(c) $C' =$ the $x-$ axis.
(d) $D' = \{(1/n, 0) \in \mathbb{R}^2 \mid n \text{ is a nonzero integer }\} \cup \{0\}$. ◇

Solution. (a) All points are isolated. If $(m, n) \in A$, then $D((m, n), 1/2) \cap A = \{(m, n)\}$. There can be no accumulation points. $A' = \emptyset$.

(b) As we have seen before, there are rational numbers in every short interval of the real line. So if (x, y) is any point in $\mathbb{R}^2$ and $\varepsilon > 0$, there are rational points closer than ε to (x, y) which are not equal to (x, y). We need only select rational numbers s and t with $0 < |s - x| < \varepsilon/\sqrt{2}$ and $0 < |t - y| < \varepsilon/\sqrt{2}$.

(c) As m and n run over the integers (with n not 0), the fraction m/n runs over all of $\mathbb{Q}$. As n increases, we get more and more points along horizontals coming close to the $x-$ axis. The result is that all points on the $x-$ axis are accumulation points. No others are. Consider Figure 2-10. So $C' = \{(x, 0) \mid x \in \mathbb{R}\}$.

(d) If we let $m = n$, we see that all of the points $1/2n$ for integer n are in D. Since these converge to 0, this must be an accumulation point. If we hold n fixed and let $m \to \infty$, we find that $1/n + 1/m \to 1/n$. So $1/n$ is an accumulation point. For m and n large, $1/n + 1/m$ moves away from all other points, so there are no other accumulation points. $D' = \{1/n \mid n \text{ is a nonzero integer }\} \cup \{0\}$. ♦

◇ **2.5-4.** (a) For $A \subset \mathbb{R}^n$, show that $\text{cl}(A) \setminus A$ consists entirely of accumulation points of A.

(b) Need it be all of them?

Suggestion. Establish first the following general observation: If A and B are sets then $(A \cup B) \setminus A \subseteq B$. Apply this with $B = A'$. For part (b) the answer is "'No". What might happen if A is a closed set? ◇

Solution. (a) From Proposition 2.5.2 we know that $\text{cl}(A) = A \cup A'$. So $\text{cl}(A) \setminus A = (A \cup A') \setminus A$.
We establish the following general observation:

Lemma. *If A and B are sets then $(A \cup B) \setminus A \subseteq B$.*

Proof: Suppose $x \in (A \cup B) \setminus A$. Then x must either be in A or B since it is in $A \cup B$. But it is not in A since the points of A have been deleted. Thus x must be in B.
Apply this with $B = A'$ to obtain

$$\text{cl}(A) \setminus A = (A \cup A') \setminus A \subseteq A'$$

as desired.

(b) The answer is "No". Some or all of the accumulation points might be in the set A. If A is closed they all are. For example, if $A = [0, 1] \subseteq \mathbb{R}$, then $A' = A = \text{cl}(A) = [0, 1]$, but $\text{cl}(A) \setminus A = \emptyset$.
Challenge: For what sets is it true that $\text{cl}(A) \setminus A = A'$? ♦

◇ **2.5-5.** In a general metric space M, let $A \subset D(x, r)$ for some $x \in M$ and $r > 0$. Show that $\text{cl}(A) \subset B(x, r) = \{y \in M \mid d(x, y) \leq r\}$.

Sketch. The set $B(x, r) = M \setminus \{y \mid d(y, x) > r\}$ is closed and $A \subseteq B(x, r)$. So $\text{cl}(A) \subseteq B(x, r)$. ◇

Solution. We know that the set $U = \{y \mid d(y, x) > r\}$ is open so that $B = \{y \mid d(y, x) \leq r\}$ is closed. (See the solution to Exercise 2.3-6 or Example 2.3.4 and Figure 2.3-3 of the text for essentially the same thing in $\mathbb{R}^2$.) We certainly have $A \subseteq D(x, r) = \{y \mid d(y, x) < r\} \subseteq B(x, r)$ since $d(y, x) < r \implies d(y, x) \leq r$. So $A \subseteq B$. We have that B is a closed set with $A \subseteq B$, so $\text{cl}(A) \subseteq B$ as claimed. ♦

◇ **2.6-5.** Let $A \subset \mathbb{R}$ be bounded and nonempty and let $x = \sup(A)$. Is $x \in \text{bd}(A)$?

Answer. Yes. ◇

Solution. Let $\varepsilon > 0$. Since $x = \sup A$, there must be an element y in A with $x - \varepsilon < x \leq x$. Every short interval around x contains points of A, so $x \in \mathrm{cl}(A)$. On the other hand, the upper half of such an interval, $]x, x+\varepsilon[$ consists entirely of points in $\mathbb{R} \setminus A$ since x is an upper bound for A. Thus $x \in \mathrm{cl}(\mathbb{R} \setminus A)$. Thus $x \in \mathrm{cl}(A) \cap \mathrm{cl}(\mathbb{R} \setminus A) = \mathrm{bd}(A)$ as desired. ♦

◇ **2.6-6.** Prove that the boundary of a set in $\mathbb{R}^2$ with the standard metric is the same as it would be with the taxicab metric.

Suggestion. Review Exercise 2.1-6. ◇

Solution. From Exercise 2.1-6, we know that the open sets are the same for these metrics. Since the closed sets are just the complements of the open sets, the closed sets are also the same in the two metrics. The closure of a set is just the intersection of all the closed sets which contain it, so the closure of a set is the same whichever metric we use. The boundary of a set is the intersection of the closure of the set with the closure of its complement. Since the closures are the same, the boundaries are the same. ♦

◇ **2.7-2.** Let $x_n \to x$ in $\mathbb{R}^m$. Show that $A = \{x_n \mid n = 1, 2, \ldots\} \cup \{x\}$ is closed.

Sketch. Let $B = \{x_1, x_2, \ldots\}$, and show that $A = \mathrm{cl}(B)$. ◇

Solution. Let $B = \{x_1, x_2, \ldots\}$. We are given that $x_n \to x$, so for each $\varepsilon > 0$, $x_n \in D(x, \varepsilon)$ for large enough n. Thus $x \in \mathrm{cl}(B)$. On the other hand, if $y \in \mathrm{cl}(B)$, then either $x \in B$ or there is a subsequence which converges to y. To see this suppose y is not in B. There is an index n_1 with $\| y - x_{n_1} \| < 1$. Since y is not among the x_n, there is an index n_2, necessarily larger than n_1, such that

$$\| y - x_{n_2} \| < \min(\| y - x_1 \|, \ldots, \| y - x_{n_1} \|)/2.$$

Continuing inductively, we get indices

$$n_1 < n_2 < n_3 < \ldots$$

such that

$$\| y - x_{n_k} \| < \min\left(\| y - x_1 \|, \ldots, \left\| y - x_{n_{k-1}} \right\| \right)/k \leq \| y - x_1 \|/k \to 0.$$

So this subsequence converges to y. But it is a subsequence of a sequence which converges to x, so we must have $y = x$. Thus the only points which can be in $\mathrm{cl}(B)$ are the points x_n and x. So $\mathrm{cl}(B) = B \cup \{x\} = \{x_n \mid n = 1, 2, 3, \ldots\} \cup \{x\}$. Thus this set is closed since the closure of any set is closed. ♦

◇ **2.8-5.** Suppose that a metric space M has the property that every bounded sequence has at least one cluster point. Show that M is complete.

Sketch. If $\langle x_n \rangle_1^\infty$ is a Cauchy sequence it is bounded. It has a cluster points and hence a convergent subsequence. So the whole sequence converges. (Why?) ◇

Solution. To show that M is complete we need to show that every cauchy sequence in M converges to a limit in M. So let $\langle x_n \rangle_1^\infty$ be a Cauchy sequence in M. By Proposition 2.8.4(ii), the sequence is bounded. By hypothesis it must have at least one cluster point x. By 2.8.7(ii) there is a subsequence which converges to x. Finally, by Proposition 2.8.4(iii), the whole sequence converges to x. Every Cauchy sequence in M converges to a point in M, so M is complete. ◆

◇ **2.9-4.** Test for convergence $\displaystyle\sum_{n=3}^{\infty} \frac{2^n + n}{3^n - n}$.

Suggestion. Try the ratio comparison test. ◇

Solution. The idea of the solution is that for large n, the exponentials 2^n and 3^n dominate the n terms and the series should behave much like the series $\sum(2^n/3^n) = \sum(2/3)^n$. We know this last series converges since it is a geometric series with ratio less than 1. So we conjecture that our original series converges. To establish this we will use the ratio comparison test. The dominance mentioned is embodied in the following observation.

Lemma. *If* $r > 1$, *then* $\displaystyle\lim_{n\to\infty} \frac{n}{r^n} = 0$.

Proof: Let $\varepsilon > 0$. Then

$$\begin{aligned} \frac{n}{r^n} < \varepsilon &\iff n < \varepsilon r^n \\ &\iff \log n < \log \varepsilon + n \log r \\ &\iff \log r > \frac{\log n - \log \varepsilon}{n}. \end{aligned}$$

The left side of the last inequality is positive since $r > 1$, and the right side tends to 0 as $n \to \infty$ by L'Hôpital's Rule. So the inequality is valid for large n, and we have our limit as claimed.

Now to solve the problem, let $a_n = \dfrac{2^n + n}{3^n - n}$ and $b_n = \dfrac{2^n}{3^n}$. Then

$$\frac{a_n}{b_n} = \frac{2^n + n}{3^n - n}\frac{3^n}{2^n} = \frac{1 + (n/2^n)}{1 - (n/3^n)} \to \frac{1+0}{1-0} = 1$$

By the ratio comparison test, the series $\displaystyle\sum_{n=3}^{\infty} \frac{2^n + n}{3^n - n}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{2^n}{3^n} = \sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$ either both converge or both diverge. We know the latter converges since it is a geometric series with ratio less than 1. So our series converges also. ◆

◇ **2.9-5.** Test for convergence $\sum_{n=0}^{\infty} \frac{n!}{3^n}$.

Answer. Does not converge. ◇

Solution. We know that $3^n/n!$ converges to 0 as $n \to \infty$ (Exercise 1.2-2). So $n!/3^n \to \infty$. In particular, the terms do not converge to 0. According to Exercise 2.9-3, the series cannot converge.

To avoid reference to §1.2, one can simply note that if $n \geq 3$, then $n!/3^n \geq 2/9$. The terms remain larger than 2/9 and the partial sums must become arbitrarily large. The sum diverges to $+\infty$. ◆

================================

◇ **2E-7.** Let U be open in a metric space M. Show that $U = \text{cl}(U) \setminus \text{bd}(U)$. Is this true for every set in M?

Answer. It is not true for every subset of M. Try $U = [0, 1] \subseteq \mathbb{R}$. ◇

Solution. Let U be an open subset of a metric space M and suppose that $x \in U$. Since U is open, $M \setminus U$ is closed. So $\text{cl}(M \setminus U) = M \setminus U$. In particular, x is not in $\text{cl}(M \setminus U)$, so it is not in $\text{bd}(U) = \text{cl}(M \setminus U) \cap \text{cl}(U)$. On the other hand, since $U \subseteq \text{cl}(A)$, we do have $x \in \text{cl}(U)$. Thus $x \in \text{cl}(U) \setminus \text{bd}(U)$. This shows that $U \subseteq \text{cl}(U) \setminus \text{bd}(U)$.

Now suppose $x \in \text{cl}(U) \setminus \text{bd}(U)$. We want to show that x must be in U. Since it is in the closure of U but not in the boundary of U, it must not be in the closure of the complement of U. But U is open, so the complement of U is closed and so equal to its closure. Thus x is not in the complement of U and so it must be in U.

We have proved inclusion in both directions, so the sets are equal as claimed.

The equality is not true for every set. Consider for example, $M = \mathbb{R}$ and $U = [0, 1]$. Since U is closed, $\text{cl}(U) = U = [0, 1]$. But the boundary of U is the two point set $\{0, 1\}$. So $\text{cl}(U) \setminus \text{bd}(U)$ is the open interval $]0, 1[$.

A good discussion question: Does this characterize open sets? Is it possible that U is open if and only if $U = \text{cl}(U) \setminus \text{bd}(U)$? ◆

◇ **2E-9.** Show that

(a) $\text{int}B = B \setminus \text{bd}B$, and
(b) $\text{cl}(A) = M \setminus \text{int}(M \setminus A)$.

Answer. Use Proposition 2.6.2 to get (a). Then use (a) to get (b). ◊

Solution. (a) Let B be a subset of a metric space M, and suppose that $x \in \text{int}(B)$. Then there is an $\varepsilon > 0$ such that $x \in D(x, \varepsilon) \subseteq B$. (See Exercise 2E-5.) This disk is contained in B and so cannot intersect $M \setminus B$. By Proposition 2.6.2, x cannot be in $\text{bd}(B)$. On the other hand, since $\text{int}(B) \subseteq B$, x is certainly in B. Thus $x \in B \setminus \text{bd}(B)$. This shows that $\text{int}(B) \subseteq B \setminus \text{bd}(B)$.

Now suppose that $x \in B \setminus \text{bd}(B)$. We want to show that x must be an interior point of B. Since $x \in B$, every disk centered at x intersects B (at least at x if nowhere else). If all such disks also intersected $M \setminus B$, then x would be in $\text{bd}(B)$ by 2.6.2. But we have assumed that x is not in $\text{bd}(B)$. So there must be an $\varepsilon > 0$ such that $D(x, \varepsilon) \cap (M \setminus B) = \emptyset$. This forces $x \in D(x, \varepsilon) \subseteq B$. So $x \in \text{int}(B)$. (See Exercise 2E-5.) This holds for every such x, so $B \setminus \text{bd}(B) \subseteq \text{int}(B)$.

We have proved inclusion in both directions, so $\text{int}(B) = B \setminus \text{bd}(B)$ as claimed.

(b) Let A be a subset of a metric space M and set $B = M \setminus A$. So $A = M \setminus B$. Using the result of part (a) we can compute:

$$\begin{aligned} M \setminus \text{int}(M \setminus A) &= M \setminus \text{int}(B) = M \setminus (B \setminus \text{bd}(B)) \\ &= M \setminus \left[B \cap (M \setminus \text{bd}(B))\right] \\ &= [M \setminus B] \cup [M \setminus (M \setminus \text{bd}(B))] \\ &= A \cup \text{bd}(B) = A \cup \text{bd}(M \setminus A) = A \cup \text{bd}(A). \end{aligned}$$

Since $\text{bd}(M \setminus A) = \text{cl}(M \setminus A) \cap \text{cl}(A) = \text{cl}(A) \cap \text{cl}(M \setminus A) = \text{bd}(A)$. We are done as soon as we establish the following:

Proposition. *If A is a subset of a metric space M, then* $\text{cl}(A) = A \cup \text{bd}(A)$.

Proof: First of all, $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(M \setminus A) \subseteq \text{cl}(A)$, and $A \subseteq \text{cl}(A)$. So $A \cup \text{bd}(A) \subseteq \text{cl}(A)$. For the opposite inclusion, suppose that $x \in \text{cl}(A)$. If $x \in A$, then it is certainly in $A \cup \text{bd}(A)$. If x is not in A, then it is in $M \setminus A$ and so in $\text{cl}(M \setminus A)$. Since we have assumed that it is in $\text{cl}(A)$, we have $x \in \text{cl}(A) \cap \text{cl}(M \setminus A) = \text{bd}(A) \subset A \cup \text{bd}(A)$. In either case, $x \in A \cup \text{bd}(A)$. So $\text{cl}(A) \subseteq A \cup \text{bd}(A)$. We have inclusion in both directions, so the sets are equal as claimed. ♦

◊ **2E-10.** Determine which of the following statements are true.

(a) $\text{int}(\text{cl}(A)) = \text{int}(A)$.
(b) $\text{cl}(A) \cap A = A$.
(c) $\text{cl}(\text{int}(A)) = A$.

(d) $\text{bd}(\text{cl}(A)) = \text{bd}(A)$.

(e) If A is open, then $\text{bd}(A) \subset M \setminus A$.

Answer. (a) May be false.

(b) True.

(c) May be false.

(d) May be false.

(e) True. ◊

Solution. (a) The equality $\text{int}(\text{cl}(A)) = \text{int}(A)$ is not always true. Consider the example $A = [-1, 1] \setminus \{0\} \subseteq \mathbb{R}$. Then $\text{cl}(A)$ is the closed interval $[-1, 1]$, and $\text{int}(\text{cl}(A))$ is the open interval $]-1, 1[$. But $\text{int}(A)$ is the open interval with zero deleted. $\text{int}(A) =]-1, 1[\setminus \{0\}$.

(b) True: Since $A \subseteq \text{cl}(A)$, we always have $\text{cl}(A) \cap A = A$.

(c) The proposed equality, $\text{cl}(\text{int}(A)) = A$, is not always true. Consider the example of a one point set with the usual metric on $\mathbb{R}$. Take $A = \{0\} \subseteq \mathbb{R}$. Then $\int A = \emptyset$. So $\text{cl}(\text{int}(A)) = \emptyset$. But A is not empty.

(d) The proposed equality, $\text{bd}(\text{cl}(A)) = \text{bd}(A)$, is not always true. Consider the same example as in part (a). $A = [-1, 1] \setminus \{0\} \subseteq \mathbb{R}$. Then $\text{cl}(A)$ is the closed interval $[-1, 1]$ and $\text{bd}(\text{cl}(A))$ is the two point set $\{-1, 1\}$. But $\text{bd}(A)$ is the three point set $\{-1, 0, 1\}$.

(e) The proposed inclusion, $\text{bd}(A) \subseteq M \setminus A$, is true if A is an open subset of the metric space M. The set A is open, so its complement, $M \setminus A$, is closed. Thus

$$\text{bd}(A) = \text{cl}(A) \cap \text{cl}(M \setminus A) = \text{cl}(A) \cap (M \setminus A) \subseteq M \setminus A$$

as claimed. ♦

◇ **2E-12.** Prove the following properties for subsets A and B of a metric space:

(a) $\text{int}(\text{int}(A)) = \text{int}(A)$.
(b) $\text{int}(A \cup B) \supset \text{int}(A) \cup \text{int}(B)$.
(c) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

Solution. (a) For any set C we know that $\text{int}(C) \subseteq C$, so, in particular, $\text{int}(\text{int}(A)) \subseteq \text{int}(A)$. For the other direction, suppose $x \in \text{int}(A)$. Then there is an open set U such that $x \in U \subseteq A$. If $y \in U$, then $y \in U \subseteq A$, so $y \in \text{int}(A)$. Thus $U \subseteq \text{int}(A)$. We have $x \in U \subseteq \text{int}(A)$, so $x \in \text{int}(\text{int}(A))$. This shows that $\text{int}(A) \subseteq \text{int}(\text{int}(A))$. We have inclusion in both directions, so $\text{int}(A) = \text{int}(\text{int}(A))$ as claimed.
Notice that in the middle of the argument just given, we established:

Proposition. *If U is open and $U \subseteq A$, then $U \subseteq \text{int}(A)$.*

This was Exercise 2E-3.

(b) Suppose $x \in \text{int}(A) \cup \text{int}(B)$. Then $x \in \text{int}(A)$ or $x \in \text{int}(B)$.

CASE 1: If $x \in \text{int}(A)$, then there is an open set U with $x \in U \subseteq A \subseteq A \cup B$. So $x \in \text{int}(A \cup B)$.

CASE 2: If $x \in \text{int}(B)$, then there is an open set U with $x \in U \subseteq B \subseteq A \cup B$. So $x \in \text{int}(A \cup B)$.

Since at least one of cases (1) or (2) must hold, we know that $x \in \text{int}(A \cup B)$. This holds for every x in $\text{int}(A) \cup \text{int}(B)$, so $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$ as claimed.

The inclusion just established could be proper. Take as subsets of the metric space $\mathbb{R}$ the "punctured" open interval $A =]-1, 1[\setminus \{0\}$, and the one point set $B = \{0\}$. Then A is open, so $\text{int}(A) = A$, and $\text{int}(B) = \emptyset$. So $\text{int}(A) \cup \text{int}(B) = A$. But $A \cup B$ is the unbroken open interval $]-1, 1[$. So $\text{int}(A \cup B) =]-1, 1[$. We see that $0 \in \text{int}(A \cup B) \setminus (\text{int}(A) \cup \text{int}(B))$.

(c) This is the same as Exercise 2.2-4. ♦

◇ **2E-13.** Show that $\text{cl}(A) = A \cup \text{bd}(A)$.

Suggestion. Compare with Exercise 2E-9.

Solution. First of all, $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(M \setminus A) \subseteq \text{cl}(A)$, and $A \subseteq \text{cl}(A)$. So $A \cup \text{bd}(A) \subseteq \text{cl}(A)$. For the opposite inclusion, suppose that $x \in \text{cl}(A)$. If $x \in A$, then it is certainly in $A \cup \text{bd}(A)$. If x is not in A, then it is in $M \setminus A$ and so in $\text{cl}(M \setminus A)$. Since we have assumed that it is in $\text{cl}(A)$, we have $x \in \text{cl}(A) \cap \text{cl}(M \setminus A) = \text{bd}(A) \subset A \cup \text{bd}(A)$. In either case, $x \in A \cup \text{bd}(A)$. So $\text{cl}(A) \subseteq A \cup \text{bd}(A)$. We have inclusion in both directions, so $\text{cl}(A) = A \cup \text{bd}(A)$ as claimed. ♦

◇ **2E-14.** Prove the following for subsets of a metric space M:

(a) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(b) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
(c) $\mathrm{cl}(A \cap B) \subset \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Solution. Let's make explicit a few lemmas about closure.

Lemma. *Suppose A, B, and C are subsets of a metric space M.*

(1) $\mathrm{cl}(A)$ *is a closed set.*
(2) $A \subseteq \mathrm{cl}(A)$.
(3) If C is closed and $A \subseteq C$, then $\mathrm{cl}(A) \subseteq C$.
(4) If $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
(5) A is closed if and only if $A = \mathrm{cl}(A)$.

Proof: Since $\mathrm{cl}(A)$ is defined as the intersection of all closed subsets of M which contain A and the intersection of any family of closed sets is closed, (Proposition 2.3.2(ii)), we see that $\mathrm{cl}(A)$ must be a closed set. Since A is contained in each of the sets being intersected, it is contained in the intersection which is $\mathrm{cl}(A)$. For part (3), if C is closed and $A \subseteq C$, then C is one of the sets being intersected to obtain $\mathrm{cl}(A)$. So $\mathrm{cl}(A) \subseteq C$.

For part (4), if $A \subseteq B$, then we know from part (2) that $A \subseteq B \subseteq \mathrm{cl}(B)$, and from part (1) that $\mathrm{cl}(B)$ is a closed set. So $\mathrm{cl}(B)$ is a closed set which contains A. By part (3) we conclude that $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ as desired.

Finally, if $A = \mathrm{cl}(A)$, then it is closed by part (1). We always have $A \subseteq \mathrm{cl}(A)$ by part (2). Of course, $A \subseteq A$, so if A is closed, then we also have $\mathrm{cl}(A) \subseteq A$ by part (3). With inclusion in both directions, we conclude that $A = \mathrm{cl}(A)$ as claimed.

(a) If A is any subset of a metric space M, then $\mathrm{cl}(A)$ is a closed set by part (1) of the lemma. So $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ by part (5).

(b) If A and B are subsets of a metric space M, then $A \subseteq A \cup B$ and $B \subseteq A \cup B$. So, by part (4) of the lemma, $\mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup B)$ and $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. So $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. On the other hand, using part (2) gives $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$. and $B \subseteq \mathrm{cl}(B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

So $A \cup B \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Each of the sets $\mathrm{cl}(A)$ and $\mathrm{cl}(B)$ is closed, so their union is also. Thus $\mathrm{cl}(A) \cup \mathrm{cl}(B)$ is a closed set containing $A \cup B$. By part (3), we have $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$. We have inclusion in both directions, so $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ as claimed.

(c) If A and B are subsets of a metric space M, then $A \cap B \subseteq A$ and $A \cap B \subseteq B$. By part (3) of the lemma we have $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(B)$. So $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$ as claimed.

Note: The inclusion established in part (b) might be proper. Consider, for example, the open intervals $A =]0,1[$ and $B =]1,2[$ as subsets of $\mathbb{R}$. Then $\mathrm{cl}(A \cap B) = \mathrm{cl}(\emptyset) = \emptyset$. But $\mathrm{cl}(A) \cap \mathrm{cl}(B) = [0,1] \cap [1,2] = \{1\}$. ♦

◇ **2E-15.** Prove the following for subsets of a metric space M:

(a) $\mathrm{bd}(A) = \mathrm{bd}(M \setminus A)$.
(b) $\mathrm{bd}(\mathrm{bd}(A)) \subset \mathrm{bd}(A)$.
(c) $\mathrm{bd}(A \cup B) \subset \mathrm{bd}(A) \cup \mathrm{bd}(B) \subset \mathrm{bd}(A \cup B) \cup A \cup B$.
(d) $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) = \mathrm{bd}(\mathrm{bd}(A))$.

Sketch. (a) Use $M \setminus (M \setminus A) = A$ and the definition of boundary.
(b) Use the fact that $\mathrm{bd}(A)$ is closed. (Why?)
(c) The facts $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ from Exercise 2E-14 are useful.
(d) One approach is to show that $\mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))) = M$ or, equivalently, that $\mathrm{int}(\mathrm{bd}(\mathrm{bd}(A))) = \emptyset$, and use that to compute $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A)))$. ◇

Solution. (a) If A is a subset of a metric space M, we can compute

$$\begin{aligned}\mathrm{bd}(M \setminus A) &= \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus (M \setminus A)) = \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(A) \\ &= \mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A) = \mathrm{bd}(A)\end{aligned}$$

as claimed.

(b) Since $\mathrm{bd}(A)$ is the intersection of $\mathrm{cl}(A)$ and $\mathrm{cl}(M \setminus A)$, both of which are closed, it is closed. In particular, $\mathrm{cl}(\mathrm{bd}(A)) = \mathrm{bd}(A)$, and we have

$$\mathrm{bd}(\mathrm{bd}(A)) = \mathrm{cl}(\mathrm{bd}(A)) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A)) = \mathrm{bd}(A) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A)).$$

Since $\mathrm{bd}(\mathrm{bd}(A))$ is the intersection of $\mathrm{bd}(A)$ with something else, we have $\mathrm{bd}(\mathrm{bd}(A)) \subseteq \mathrm{bd}(A)$ as claimed.

(c) A key to part (c) are the observations that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for any subsets A and B of a metric space

M. (See Exercise 2E-14.) Using them we can compute

$$\begin{aligned}
\mathrm{bd}(A \cup B) &= \mathrm{cl}(A \cup B) \cap \mathrm{cl}(M \setminus (A \cup B)) \\
&= \mathrm{cl}(A \cup B) \cap \mathrm{cl}((M \setminus A) \cap (M \setminus B)) \\
&\subseteq \mathrm{cl}(A \cup B) \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B) \\
&= [\mathrm{cl}(A) \cup \mathrm{cl}(B)] \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B) \\
&= [\mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B)] \\
&\qquad \cup [\mathrm{cl}(B) \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B)] \\
&\subseteq [\mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A)] \cup [\mathrm{cl}(B) \cap \mathrm{cl}(M \setminus B)] = \mathrm{bd}(A) \cup \mathrm{bd}(B).
\end{aligned}$$

This is the first inclusion claimed. Now suppose $x \in \mathrm{bd}(A) \cup \mathrm{bd}(B)$. Then $x \in \mathrm{cl}(A)$ or $x \in \mathrm{cl}(B)$. So $x \in \mathrm{cl}(A) \cup \mathrm{cl}(B) = \mathrm{cl}(A \cup B)$. If x is in neither A nor B, then $x \in (M \setminus A) \cap (M \setminus B) = M \setminus (A \cup B) \subseteq \mathrm{cl}((M \setminus (A \cup B))$. Since we also have $x \in \mathrm{cl}(A \cup B)$, this puts x in $\mathrm{bd}(A \cup B)$. So x must be in at least one of the three sets A, B, or $\mathrm{bd}(A \cup B)$. That is, $\mathrm{bd}(A) \cup \mathrm{bd}(B) \subseteq \mathrm{bd}(A \cup B) \cup A \cup B$ as claimed.

(d) From part (a) we know that $\mathrm{bd}(\mathrm{bd}(C)) \subseteq \mathrm{bd}(C)$ for every set C. So, in particular, $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) \subseteq \mathrm{bd}(\mathrm{bd}(A))$. But now we want equality:

$$\begin{aligned}
\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) &= \mathrm{cl}(\mathrm{bd}(\mathrm{bd}(A))) \cap \mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))) \\
&= \mathrm{bd}(\mathrm{bd}(A)) \cap \mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))).
\end{aligned}$$

But

$$\begin{aligned}
\mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))) &= \mathrm{cl}(M \setminus (\mathrm{cl}(\mathrm{bd}(A)) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A)))) \\
&= \mathrm{cl}(M \setminus (\mathrm{bd}(A) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A)))) \\
&= \mathrm{cl}((M \setminus \mathrm{bd}(A)) \cup (M \setminus \mathrm{cl}(M \setminus \mathrm{bd}(A)))) \\
&= \mathrm{cl}(M \setminus \mathrm{bd}(A)) \cup \mathrm{cl}(M \setminus \mathrm{cl}(M \setminus \mathrm{bd}(A))) \\
&\subseteq \mathrm{cl}(M \setminus \mathrm{bd}(A)) \cup \mathrm{cl}(M \setminus (M \setminus \mathrm{bd}(A))) \\
&\subseteq (M \setminus \mathrm{bd}(A)) \cup (M \setminus (M \setminus \mathrm{bd}(A))) = M.
\end{aligned}$$

Combining the last two displays gives

$$\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) = \mathrm{bd}(\mathrm{bd}(A)) \cap M = \mathrm{bd}(\mathrm{bd}(A))$$

as claimed.

Remark: The identity in the next to last display is equivalent to the assertion that

$$\mathrm{int}(\mathrm{bd}(\mathrm{bd}(A))) = \emptyset. \qquad \blacklozenge$$

◇ **2E-20.** For a set A in a metric space M and $x \in M$, let

$$d(x, A) = \inf\{d(x, y) \mid y \in A\},$$

and for $\varepsilon > 0$, let $D(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$.

(a) Show that $D(A, \varepsilon)$ is open.

(b) Let $A \subset M$ and $N_\varepsilon = \{x \in M \mid d(x, A) \leq \varepsilon\}$, where $\varepsilon > 0$. Show that N_ε is closed and that A is closed iff $A = \cap \{N_\varepsilon \mid \varepsilon > 0\}$.

Suggestion. For (a), show that $D(A, \epsilon)$ is a union of open disks. For the first part of (b), consider convergent sequences in $N(A, \varepsilon)$ and their limits in M. ◇

Solution. (a) First suppose $x \in D(A, \varepsilon)$. Then $d(x, A) = \inf\{d(x, y) \mid y \in A\} = r < \varepsilon$. So there is a point $y \in A$ with $r \leq d(x, y) < \varepsilon$. Thus $x \in D(y, \varepsilon)$. This can be done for each $x \in D(A, \varepsilon)$. We conclude that

$$D(A, \varepsilon) \subseteq \bigcup_{y \in A} D(y, \varepsilon).$$

Conversely, if there is a y in A with $x \in D(y, \varepsilon)$, then $d(x, y) < \varepsilon$. So $d(x, A) = \inf\{d(x, y) \mid y \in A\} < \varepsilon$, and $x \in D(A, \varepsilon)$. This proves inclusion in the other direction. We conclude that

$$D(A, \varepsilon) = \bigcup_{y \in A} D(y, \varepsilon).$$

Each of the disks $D(y, \varepsilon)$ is open by Proposition 2.1.2, so their union, $D(A, \varepsilon)$ is also open by Proposition 2.1.3(ii).

(b) To show that $N(A, \varepsilon)$ is closed we will show that it contains the limits of all convergent sequences in it. Suppose $\langle x_k \rangle_1^\infty$ is a sequence in $N(A, \varepsilon)$ and that $x_k \to x \in M$. Since each $x_k \in N(A, \varepsilon)$, we have $d(x_k, A) \leq \varepsilon < \varepsilon + (1/k)$. So there are points y_k in A with $d(x_k, y_k) < \varepsilon + (1/k)$. Since $x_k \to x$, we know that $d(x, x_k) \to 0$, and can compute

$$d(x, y_k) \leq d(x, x_k) + d(x_k, y_k) < d(x, x_k) + \varepsilon + \frac{1}{k} \to \varepsilon \quad \text{as } k \to \infty.$$

Thus

$$d(x, A) = \inf\{d(x, y) \mid y \in A\} \leq \varepsilon.$$

So $x \in N(A, \varepsilon)$. We have shown that if $\langle x_k \rangle_1^\infty$ is a sequence in $N(A, \varepsilon)$ and $x_k \to x \in M$, then $x \in N(A, \varepsilon)$. So $N(A, \varepsilon)$ is a closed subset of M by Proposition 2.7.6(i).

We have just shown that each of the sets $N(A, \varepsilon)$ is closed, and we know that the intersection of any family of closed subsets of M is closed. So if $A = \bigcap_{\varepsilon > 0} N(A, \varepsilon)$, then A is closed.

Conversely, if A is closed and $y \in M \setminus A$, then there is an $r > 0$ such that $D(y, r) \subset M \setminus A$ since the latter set is open. Thus y is not in $N(A, r/2)$. So y is not in $\bigcap_{\varepsilon > 0} N(A, \varepsilon)$. This establishes the opposite inclusion $\bigcap_{\varepsilon > 0} N(A, \varepsilon) \subseteq A$.

If A is closed, we have inclusion in both directions, so $A = \bigcap_{\varepsilon > 0} N(A, \varepsilon)$ as claimed. ♦

◇ **2E-25.** Prove that a set $A \subset M$ is open iff we can write A as the union of some family of ε-disks.

Sketch. Since ε–disks are open, so is any union of them. Conversely, if A is open and $x \in A$, there is an $\varepsilon_x > 0$ with $x \in D(x, \varepsilon_x) \subseteq A$. So $A = \bigcup_{x \in A} D(x, \varepsilon_x)$. ◊

Solution. To say that A is a union of ε–disks is to say that there is a set of points $\{x_\beta \mid \beta \in B\} \subseteq A$ and a set of positive radii $\{r_\beta \mid \beta \in B\}$ such that $A = \bigcup_{\beta \in B} D(x_\beta, r_\beta)$. ($B$ is just any convenient index set for listing these things.) We know from Proposition 2.1.2 that each of the disks $D(x_\beta, r_\beta)$ is open. By 2.1.3(ii), the union of any family of open subsets of M is open. So A must be open.

For the converse, suppose A is an open subset of A. Then for each x in A, there is a radius $r_x > 0$ such that $D(x, r_x) \subseteq A$. Since $x \in D(x, r_x) \subseteq A$, we have

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} D(x, r_x) \subseteq A.$$

So we must have $A = \bigcup_{x \in A} D(x, r_x)$, a union of open disks, as required. ♦

◇ **2E-31.** Let A' denote the set of accumulation points of a set A. Prove that A' is closed. Is $(A')' = A'$ for all A?

Sketch. What needs to be done is to show that an accumulation point of A' must be an accumulation point of A. $(A')'$ need not be equal to A'. Consider $A = \{1/2, 1/3, \ldots\}$. ◊

Solution. To show that A' is closed, we show that it contains all of its accumulation points. That is, $(A')' \subseteq A'$. Suppose x is an accumulation point of A', and let U be an open set containing x. Then U contains a point y in A' with y not equal to x. Let $V = U \setminus \{x\}$. Then V is an open set containing y. Since $y \in A'$, there is a point z in $V \cap A$ with z not equal to y. Since x is not in V, we also know that z is not equal to x. Since $V \subseteq U$, we know that $z \in U$. Every neighborhood U of x contains a point z of A which is not equal to x. So x is an accumulation point of A. This works for every x in A'. So $(A')' \subseteq A'$. Since the set A' contains all of its accumulation points, it is closed as claimed.

Although we now know that $(A')' \subseteq A'$ for all subsets of a metric space M, the inclusion might be proper. Consider $A = \{1, 1/2, 1/3 \ldots\} \subseteq \mathbb{R}$. Then $A' = \{0\}$. and $(A')' = \emptyset$. ♦

◇ **2E-35.** Show that any family of disjoint nonempty open sets of real numbers is countable.

Suggestion. Each of the sets must contain a rational number. ◊

Solution. Let $\{U_\alpha \mid \alpha \in A\}$ be any family of disjoint open subsets of $\mathbb{R}$. That is, each U_α is open, and $U_\alpha \cap U_\beta = \emptyset$ if α and β are different. Since each U_α is open, there are open intervals $]a_\alpha, b_\alpha[\subseteq U_\alpha$. We know that the rational numbers are scattered densely in $\mathbb{R}$ in the sense that if $a_\alpha < b_\alpha$ is any such pair of reals there is a rational number r_α with $r_\alpha \in]a_\alpha, b_\alpha[\subseteq U_\alpha$. Since different α and β have $U_\alpha \cap U_\beta = \emptyset$, we must have r_α and r_β different. But there are only countably many different rational numbers. So there are only countably many different sets U_α in the collection.

Remarks: (1) Without too much effort we can see that points $(r_1, r_2, \ldots, r_n)$ with rational coordinates are dense in $\mathbb{R}^n$ in the sense that any open ball $D(v, \epsilon)$ must contain such points. Furthermore, $\mathbb{Q} \times \cdots \times \mathbb{Q} = \mathbb{Q}^n \subseteq \mathbb{R}^n$ is a countable set of points in $\mathbb{R}^n$. With these facts in hand we can conclude in exactly the same way that every family of disjoint open subsets in $\mathbb{R}^n$ must be countable.

(2) If we combine the results of Exercises 2E-30 and 2E-34, we obtain

Proposition. *Each open set in $\mathbb{R}$ is the union of countably many disjoint open intervals.*

(3) This also has a generalization to $\mathbb{R}^n$. We saw that in Exercise 2E-30 that the appropriate sets to look at are not the direct generalization of intervals to "rectangles". We need to loosen this to "connected sets". These are those sets which are "all in one piece". We will find the right way to formulate this in Chapter 3.

Proposition. *Each open subset of $\mathbb{R}^n$ is the union of a countable number of disjoint connected subsets.*

The connection is that a subset of $\mathbb{R}$ is connected (all one piece) if and only if it is an interval. Here the term "interval" is intended to include half lines and the whole line. ♦

◇ **2E-37.** For $A \subset M$, a metric space, prove that

$$\mathrm{bd}(A) = [A \cap \mathrm{cl}(M \setminus A)] \cup [\mathrm{cl}(A) \setminus A].$$

Sketch. $[A \cap \mathrm{cl}(M \setminus A)] \cup [\mathrm{cl}(A) \setminus A] = [A \cup (\mathrm{cl}(A) \setminus A)] \cap [\mathrm{cl}(M \setminus A) \cup (\mathrm{cl}(A) \setminus A)] = \mathrm{cl}(A) \cap [\mathrm{cl}(M \setminus A) \cup (\mathrm{cl}(A) \setminus A)]$. Why is this $\mathrm{bd}(A)$? ◊

Solution. If $x \in \mathrm{cl}(A) \setminus A$, then it is not in A and must be in $M \setminus A$. So it is certainly in $\mathrm{cl}(M \setminus A)$. Thus $\mathrm{cl}(A) \setminus A \subseteq \mathrm{cl}(M \setminus A)$, and

$$\mathrm{cl}(M \setminus A) \cup (\mathrm{cl}(A) \setminus A) = \mathrm{cl}(M \setminus A).$$

We compute

$$\begin{aligned}
[A \cap \mathrm{cl}(M \setminus A)] \cup [\mathrm{cl}(A) \setminus A] &= [A \cup (\mathrm{cl}(A) \setminus A)] \cap [\mathrm{cl}(M \setminus A) \cup (\mathrm{cl}(A) \setminus A)] \\
&= \mathrm{cl}(A) \cap [\mathrm{cl}(M \setminus A) \cup (\mathrm{cl}(A) \setminus A)] \\
&= \mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A) \\
&= \mathrm{bd}(A)
\end{aligned}$$

as claimed♦

◇ **2E-39.** Let $S \subset \mathbb{R}$ be bounded above and below. Prove that $\sup(S) - \inf(S) = \sup\{x - y \mid x \in S \text{ and } y \in S\}$.

Suggestion. First show that $\sup(S) - \inf(S)$ is an upper bound for $\{x - y \mid x \in S \text{ and } y \in S\}$. Then consider x and y in S very close to $\sup(S)$ and $\inf(S)$. ♢

Solution. Let $T = \{x - y \mid x \in S \text{ and } y \in S\}$. If x and y are in S, we know that $x \le \sup S$ and $\inf S \le y$. So $-y \le -\inf S$, and

$$x - y \le x - \inf S \le \sup S - \inf S.$$

Thus $\sup S - \inf S$ is an upper bound for T. If $\varepsilon > 0$, there are x and y is S with $\sup S - (\varepsilon/2) < x \le \sup S$, and $\inf S \le y < \inf S + (\varepsilon/2)$. So

$$(\sup S - \inf S) - \varepsilon = \sup S - (\varepsilon/2) - (\inf S + (\varepsilon/2)) < x - y \le \sup S - \inf S.$$

Thus $(\sup S - \inf S) - \varepsilon$ is not an upper bound for T. This holds for every $\varepsilon > 0$, so $\sup S - \inf S$ is the least upper bound for T as claimed. ♦

◇ **2E-44.** A set $A \subset \mathbb{R}^n$ is said to be ***dense*** in $B \subset \mathbb{R}^n$ if $B \subset \text{cl}(A)$. If A is dense in $\mathbb{R}^n$ and U is open, prove that $A \cap U$ is dense in U. Is this true if U is not open?

Sketch. It need not be true for sets which are not open. ♢

Solution. Let A be dense in $\mathbb{R}^n$ and U is an open subset of $\mathbb{R}^n$. We want to prove that $U \subseteq \text{cl}(A \cap U)$. If $x \in U$, we need to show that every neighborhood of x intersects $A \cap U$. So, suppose V is an open set containing x. Then $x \in V \cap U$ which is an open set. So there is an $r > 0$ such that $x \in D(x, r) \subseteq U \cap V$. Since A is dense in $\mathbb{R}^n$, there is a

$$y \in A \cap D(x, r) \subseteq A \cap (V \cap U) = V \cap (A \cap U).$$

Since this can be done for every neighborhood V of x, we have $x \in \text{cl}(A \cap U)$. Since this is true for every $x \in U$, we have $U \subseteq \text{cl}(A \cap U)$. Thus $A \cap U$ is dense in U as claimed.

If the set U is not open, this can fail. Consider $U = \mathbb{Q} \subseteq \mathbb{R}$ and $A = \mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$. Each of the sets U and A is dense in $\mathbb{R}$, but $A \cap U = \emptyset$ and is dense nowhere. ♦

◇ **2E-51.** (a) If $u_n > 0$, $n = 1, 2, \ldots$, show that

$$\liminf \frac{u_{n+1}}{u_n} \leq \liminf \sqrt[n]{u_n} \leq \limsup \sqrt[n]{u_n} \leq \limsup \frac{u_{n+1}}{u_n}.$$

(b) Deduce that if $\lim(u_{n+1}/u_n) = A$, then $\limsup \sqrt[n]{u_n} = A$.

(c) Show that the converse of part (b) is false by use of the sequence $u_{2n} = u_{2n+1} = 2^{-n}$.

(d) Calculate $\limsup \sqrt[n]{n!}/n$.

Answer. (d) 1/e. ◇

Solution. (a) There are three inequalities to prove:

$$\liminf_{n\to\infty} \frac{u_{n+1}}{u_n} \leq \liminf_{n\to\infty} \sqrt[n]{u_n} \tag{1}$$

$$\liminf_{n\to\infty} \sqrt[n]{u_n} \leq \limsup_{n\to\infty} \sqrt[n]{u_n} \tag{2}$$

$$\limsup_{n\to\infty} \sqrt[n]{u_n} \leq \limsup_{n\to\infty} \frac{u_{n+1}}{u_n}. \tag{3}$$

Of these the second is true for every sequence, so we need only work on the first and third.

Proof of (1): We know that $u_n > 0$ for each n. So $u_{n+1}/u_n > 0$ and $\sqrt[n]{u_n} > 0$. Therefore $\liminf(u_{n+1}/u_n) \geq 0$ and $\liminf \sqrt[n]{u_n} \geq 0$. If $\liminf(u_{n+1}/u_n) = 0$, then inequality (1) is certainly true. So we may assume that $\liminf(u_{n+1}/u_n) > 0$. If $a < \liminf(u_{n+1}/u_n)$, we can select a number r with $a < r < \liminf(u_{n+1}/u_n)$. Then there is an integer N such that $u_{n+1}/u_n > r$ whenever $n \geq N$. Since $u_n > 0$, we have $u_{n+1} > ru_n$ for $n \geq N$. Applying this repeatedly we find

$$\begin{aligned}
u_{N+1} &\geq ru_N \\
u_{N+2} &\geq r_{N+1} \geq r^2 u_N \\
u_{N+3} &\geq r_{N+2} \geq r^3 u_N \\
&\vdots \\
u_{N+k} &\geq r_{N+k-1} \geq r^k u_N \\
&\vdots
\end{aligned}$$

So

$$\sqrt[N+k]{u_{N+k}} > \sqrt[N+k]{r^k u_N} = r^{k/(N+k)} u_N^{1/(N+k)}.$$

Now we need two facts about exponential functions: $r^x \to r$ as $x \to 1$ and $c^x \to 1$ as $x \to 0$ for positive constants r and c. Applying this to the rightmost expression in the last display, we find

$$\sqrt[N+k]{u_{N+k}} > r^{k/(N+k)} u_N^{1/(N+k)} \to r > a \qquad \text{as} \qquad k \to \infty.$$

So $\sqrt[N+k]{u_{N+k}} > a$ for sufficiently large k. That is, $\sqrt[n]{u_n} > a$ for sufficiently large n. Thus $\liminf \sqrt[n]{u_n} \geq a$. This is true for every a smaller than $\liminf u_{n+1}/u_n$. So

$$\liminf u_{n+1}/u_n \leq \liminf \sqrt[n]{u_n}$$

as claimed.

Proof of (3): The argument for the third inequality is similar to that for the first but with the inequalities reversed. If $\limsup(u_{n+1}/u_n) = +\infty$, then inequality (3) is certainly true. Therefore we may assume that $\limsup(u_{n+1}/u_n) < +\infty$. If $b > \limsup(u_{n+1}/u_n)$, we can select a number r with $b > r > \limsup(u_{n+1}/u_n)$. Then there is an integer N such that $u_{n+1}/u_n < r$ whenever $n \geq N$. Since $u_n > 0$, we have

$u_{n+1} < ru_n$ for $n \geq N$. Applying this repeatedly we find

$$\begin{aligned}
u_{N+1} &\leq ru_N \\
u_{N+2} &\leq r_{N+1} \leq r^2 u_N \\
u_{N+3} &\leq r_{N+2} \leq r^3 u_N \\
&\vdots \\
u_{N+k} &\leq r_{N+k-1} \leq r^k u_N \\
&\vdots
\end{aligned}$$

So

$$\sqrt[N+k]{u_{N+k}} \leq \sqrt[N+k]{r^k u_N} = r^{k/(N+k)} u_N^{1/(N+k)} \to r < b$$

as $k \to \infty$. So $\sqrt[N+k]{u_{N+k}} < b$ for sufficiently large k. That is, $\sqrt[n]{u_n} < b$ for sufficiently large n. Thus $\limsup \sqrt[n]{u_n} \leq b$. This is true for every b larger than $\limsup u_{n+1}/u_n$. So

$$\limsup \sqrt[n]{u_n} \leq \limsup u_{n+1}/u_n$$

as claimed.

(b) We know from Proposition 1.5.7(ix), that the limit of a sequence exists if and only if the limit inferior and the limit superior are the same and equal to that limit. Since $\lim_{n\to\infty}(u_{n+1}/u_n)$ has been assumed to exist and be equal to A, we know that

$$A = \liminf \frac{u_{n+1}}{u_n} \leq \liminf \sqrt[n]{u_n} \leq \limsup \sqrt[n]{u_n} \leq \limsup \frac{u_{n+1}}{u_n} = A.$$

So we must have

$$\liminf \sqrt[n]{u_n} = \limsup \sqrt[n]{u_n} = A.$$

So $\lim_{n\to\infty} \sqrt[n]{u_n}$ exists and is equal to A by 1.5.7(ix).

(c) If the sequence u_n is defined by $u_{2n} = u_{2n+1} = 2^{-n}$ for $n = 0, 1, 2, \ldots$, then the first few terms are

$$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \frac{1}{2^n}, \ldots.$$

The ratios of succeeding terms, u_{n+1}/u_n, are

$$1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \ldots.$$

So $\liminf u_{n+1}/u_n = 1/2$ while $\limsup u_{n+1}/u_n = 1$. But if $k = 2n$, then

$$\sqrt[k]{u_k} = \sqrt[2n]{2^{-n}} = 2^{-n/2n} = 2^{-1/2} = 1/\sqrt{2}.$$

While if $k = 2n + 1$, then

$$\sqrt[k]{u_k} = \sqrt[2n+1]{2^{-n}} = 2^{-n/(2n+1)} \to 2^{-1/2} = 1/\sqrt{2}$$

So

$$\liminf_{k\to\infty} \sqrt[k]{u_k} = \limsup_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} \sqrt[k]{u_k} = 1/\sqrt{2}.$$

So $\liminf \sqrt[k]{u_k}$ and $\limsup \sqrt[k]{u_k}$ are the same while $\liminf(u_{k+1}/u_k)$ and $\limsup(u_{k+1}/u_k)$ are different. The converse of the result of part (b) is false.

(d) Let $a_n = \sqrt[n]{n!}/n$ and $u_n = n!/n^n$. Then $\sqrt[n]{u_n} = a_n$. We compute

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e}.$$

(To get the last limit, take logarithms and use L'Hôpital's Rule to show that $(1 + (1/n))^n \to e$.) Since this limit exists, we conclude from part (b) that $\lim \sqrt[n]{u_n}$ also exists and is $1/e$. Thus $\lim_{n\to\infty} \sqrt[n]{n!}/n$ exists and is equal to $1/e$. ♦