Exercises within a Section: 1.1-2 means Section 1.1 problem #2 etc...

☞**Watch out! To save paper and spaces, some solutions may not be in the proper order. You should be able to find them.**（解答裡題目的順序可能不會照書上的順序，請大家在「該出現的地方」找不到時往後翻一下）

☞**You are required to reproduce or to paraphrase of the "Solution" (NOT the "Sketch") to a problem.** （考試寫證明的時候，是要寫 Solution 的部份，而不是 Sketch 的部份！Sketch 的部份是告訴大家證明的 idea 是什麼。）

☞**Do not skip over problems that you think are complicated. We can still ask you part of the steps in a test.**（不要跳過你覺得證明太複雜的問題，我們考試時仍可能會考你！）

◇ **1.1-2.** In a field, show that if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Solution. Recall that division is defined by $\frac{x}{y} = xy^{-1}$. We compute:

$$\begin{aligned}
\frac{a}{b} + \frac{c}{d} &= \left(\frac{a}{b} + \frac{c}{d}\right)\left((bd)(bd)^{-1}\right) && \text{multiply by one} \\
&= \left(\left(\frac{a}{b} + \frac{c}{d}\right)(bd)\right)(bd)^{-1} && \text{associative law} \\
&= ((ab^{-1} + cd^{-1})(bd))(bd)^{-1} && \text{definition of division} \\
&= ((ab^{-1})(bd) + (cd^{-1})(bd))(bd)^{-1} && \text{distributive law} \\
&= ((ab^{-1})b)d + (cd^{-1})d)b)(bd)^{-1} && \text{assoc. and comm. laws} \\
&= ((a(b^{-1}b))d + (c(d^{-1}d))b)(bd)^{-1} && \text{associative law} \\
&= ((a(1))d + (c(1))b)(bd)^{-1} && \text{reciprocals} \\
&= (ad + cb)(bd)^{-1} = (ad+bc)(bd)^{-1} && \text{property of one} \\
&= \frac{ad+bc}{bd} && \text{definition of division}
\end{aligned}$$

as claimed. ♦

◇ **1.1-5.** Give an example of a field with only three elements. Prove that it cannot be made into an ordered field.

Sketch. Let $\mathbb{F} = \{0, 1, 2\}$ with arithmetic mod 3. For example, $2 \cdot 2 = 1$ and $1 + 2 = 0$. To show it cannot be ordered, get a contradiction from (for example) $1 > 0$ so $1 + 1 = 2 > 0$ so $1 + 2 = 0 > 0$. ♢

Solution. Let $\mathbb{F} = \{0, 1, 2\}$ with arithmetic mod 3. For example, $2 \cdot 2 = 1$ and $1 + 2 = 0$. The commutative associative and distributive properties work for modular arithmetic with any base. When the base is a prime, the result is a field. In particular in arithmetic modulo 3 we have

$$1 \cdot 1 = 1 \quad \text{and} \quad 2 \cdot 2 = 1.$$

Thus 1 and 2 are their own reciprocals. Since they are the only two nonzero elements, we have a field.

To show it cannot be ordered, get a contradiction from (for example) $1 > 0$ so $1+1 = 2 > 0$ so $1+2 = 0 > 0$. We know that $0 \le 1$ in any ordered field. So $1 \le 1 + 1 = 2$ by order axiom 15. Transitivity gives $0 \le 2$. So far there is no problem. But, if we add 1 to the inequality $1 \le 2$ obtained above, we find $2 = 1 + 1 \le 2 + 1 = 0$. So $2 \le 0$. Thus $0 = 2$. If we multiply by 2, we get $0 = 0 \cdot 2 = 2 \cdot 2 = 1$. So $0 = 1$. But we know this is false. ♦

Note: The following exercise is not new, you should know how to compute the limit from first year calculus:

◇ **1.2-3.** Let $x_n = \sqrt{n^2 + 1} - n$. Compute $\lim_{n\to\infty} x_n$.

Answer. 0. ♢

Solution. Let $x_n = \sqrt{n^2 + 1} - n$, and notice that

$$(\sqrt{n^2 + 1} - n) \cdot (\sqrt{n^2 + 1} + n) = (n^2 + 1) - n^2 = 1.$$

Thus we have

$$0 \le x_n = \sqrt{n^2 + 1} - n = \frac{1}{\sqrt{n^2 + 1} + n} \le \frac{1}{\sqrt{n^2} + n} = \frac{1}{2n} \le \frac{1}{n}.$$

We know that $1/n \to 0$, so $x_n \to 0$ by the "Sandwich Lemma" 1.2.2. ♦

◇ **1.2-4.** Let x_n be a monotone increasing sequence such that $x_{n+1} - x_n \leq 1/n$. Must x_n converge?

Answer. No, not necessarily. ◇

Solution. If we put $x_1 = 1$ and suppose that $x_{n+1} = x_n + (1/n)$, then

$$
\begin{aligned}
x_2 &= 1 + \frac{1}{1} \\
x_3 &= x_2 + \frac{1}{2} = 1 + \frac{1}{1} + \frac{1}{2} \\
x_4 &= x_3 + \frac{1}{3} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\
&\vdots \\
x_{n+1} &= x_n + \frac{1}{n} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}
\end{aligned}
$$

Since we know that the harmonic series diverges to infinity, we see that the sequence $\langle x_n \rangle_1^\infty$ cannot converge. ♦

◇ **1.3-5.** Let $S \subset [0, 1]$ consist of all infinite decimal expansions $x = 0.a_1a_2a_3 \cdots$ where all but finitely many digits are 5 or 6. Find $\sup S$.

Answer. $\sup(S) = 1$. ◇

Solution. The numbers $x_n = 0.99999\ldots9995555555\ldots$ consisting of n 9's followed by infinitely many 5's are all in S. Since these come as close to 1 as we want, we must have $\sup S = 1$. ♦

◇ **1.3-4.** Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded below and define $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$. Is it true that $\inf(A + B) = \inf A + \inf B$?

Answer. Yes. ◊

Solution. First suppose $z \in A + B$, then there are points $x \in A$ and $y \in B$ with $z = x + y$. Certainly $\inf A \le x$ and $\inf B \le y$. So

$$\inf A + \inf B \le x + y = z.$$

Thus $\inf A + \inf B$ is a lower bound for the set $A + B$. So $\inf A + \inf B \le \inf(A + B)$.

To get the opposite inequality, let $\varepsilon \ge 0$. There must be points $x \in A$ and $y \in B$ with

$$\inf A \le x < \inf A + \frac{\varepsilon}{2} \qquad \text{and} \qquad \inf B \le y < \inf B + \frac{\varepsilon}{2}.$$

Since $x + y \in A + B$ we must have

$$\inf(A + B) \le x + y \le \inf A + \frac{\varepsilon}{2} + \inf B + \frac{\varepsilon}{2} = \inf A + \inf B + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we must have $\inf(A + B) \le \inf A + \inf B$.

We have inequality in both directions, so $\inf(A+B) = \inf A + \inf B$. ♦

◇ **1.7-3.** Put the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ on $\mathcal{C}([0, 1])$. Verify the Cauchy-Schwarz inequality for $f(x) = 1$ and $g(x) = x$.

Sketch. $\sqrt{\langle f, f \rangle} = 1$, $\sqrt{\langle g, g \rangle} = 1/\sqrt{3}$, and $\langle f, g \rangle = 1/2$. So $|\langle f, g \rangle| \le \sqrt{\langle f, f \rangle} \cdot \sqrt{\langle g, g \rangle}$ is true. ◊

Solution. $\sqrt{\langle f, f \rangle} = \left(\int_0^1 (f(x))^2\, dx\right)^{1/2} = \left(\int_0^1 1\, dx\right)^{1/2} = 1,$

$\sqrt{\langle g, g \rangle} = \left(\int_0^1 (g(x))^2\, dx\right)^{1/2} = \left(\int_0^1 x^2\, dx\right)^{1/2} = 1/\sqrt{3},$

and $\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx = \int_0^1 x\, dx = 1/2.$

So $|\langle f, g \rangle| \le \sqrt{\langle f, f \rangle} \cdot \sqrt{\langle g, g \rangle}$ is true. ♦

◇ **1.4-4.** Let x_n be a Cauchy sequence. Suppose that for every $\varepsilon > 0$ there is some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$. Prove that $x_n \to 0$.

Discussion. The assumption that $\langle x_n \rangle_1^\infty$ is a Cauchy sequence says that far out in the sequence, all of the terms are close to each other. The second assumption, that for every $\varepsilon > 0$ there is an $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$, says, more or less, that no matter how far out we go in the sequence there will be at least one term out beyond that point which is small. Combining these two produces the proof. If some of the points far out in the sequence are small and all of the points far out in the sequence are close together, then all of the terms far out in the sequence must be small. The technical tool used to merge the two assumptions is the triangle inequality. ◇

Solution. Let $\varepsilon > 0$. Since the sequence is a Cauchy sequence, there is an N_1 such that $|x_n - x_k| < \varepsilon/2$ whenever $n \geq N_1$ and $k \geq N_1$.

Pick $N_2 > N_1$ large enough so that $1/N_2 < \varepsilon/2$. By hypothesis there is at least one index $n > 1/(1/N_2) = N_2$ with $|x_n| < 1/N_2$.

If $k \geq N_1$, then both k and n are at least as large as N_1 and we can compute

$$|x_k| = |x_k - x_n + x_n| \leq |x_k - x_n| + |x_n| < \frac{\varepsilon}{2} + \frac{1}{N_2} < \varepsilon$$

Thus $x_k \to 0$ as claimed. ♦

◇ **1.4-5.** True or false: If x_n is a Cauchy sequence, then for n and m large enough, $d(x_{n+1}, x_{m+1}) \leq d(x_n, x_m)$.

Answer. False. ◇

Solution. If this were true, it would hold in particular with m selected as $m = n + 1$. That is, we should have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ for large enough n. But this need not be true. Consider the sequence

$$1, 0, 0, \frac{1}{2}, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{5}, \ldots .$$

Then

$$\begin{aligned} d(x_1, x_2) &= 1 \\ d(x_2, x_3) &= 0 \\ d(x_3, x_4) &= \frac{1}{2} \\ d(x_4, x_5) &= \frac{1}{2} \\ d(x_5, x_6) &= 0 \\ d(x_6, x_7) &= \frac{1}{3} \\ d(x_7, x_8) &= \frac{1}{3} \\ d(x_8, x_9) &= 0 \\ &\vdots \end{aligned}$$

This sequence converges to 0 and is certainly a Cauchy sequence. But the differences of succeeding terms keep dropping to 0 and then coming back up a bit. ♦

◇ **1.5-4.** Let $\limsup x_n = 2$. True or false: If n is large enough, then $x_n > 1.99$.

Answer. False. ◇

Solution. The largest cluster point is 2, but this need not be the only cluster point. Consider for example the sequence

$$1, 2, 1, 2, 1, 2, 1, 2, \ldots,$$

where

$$x_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases} .$$

In this example we have $x_n < 1.99$ for all odd n. ◆

◇ **1.5-5.** True or false: If $\limsup x_n = b$, then for n large enough, $x_n \le b$.

Answer. False. ◇

Solution. A counterexample is easy to give. For example, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

converges to a limit of 0. So the lim sup and the lim inf are both equal to 0. All terms of the sequence are larger than the lim sup. ◆

◇ **1.6-1.** If $\| x + y \| = \| x \| + \| y \|$, show that x and y lie on the same ray from the origin.

Sketch. If equality holds in the Cauchy-Schwarz or triangle inequality, then x and y are parallel. Expand $\| x + y \|^2 = (\| x \| + \| y \|)^2$ to obtain $\| x \| \cdot \| y \| = \langle x, y \rangle = \| x \| \cdot \| y \| \cdot \cos(\vartheta)$. Use this directly or let $u = y / \| y \|$ and $z = x - \langle x, u \rangle u$. Check that $\langle z, u \rangle = 0$ and that $\| x \|^2 = \| z \|^2 + |\langle x, u \rangle|^2 = \| x \|^2$. Conclude that $z = 0$ and $x = (\langle x, y \rangle / \| y \|^2) y$. ◇

Solution. If either x or y is 0, then they are certainly on the same ray through the origin. If y is not the zero vector, we can let u be the unit vector $u = (1/ \| y \|)\, y$. The projection of x in the direction of y is then

$v = \langle x, u \rangle \, u = (\langle x, y \rangle / \| y \|^2) y$. We claim that this is equal to x. Let $z = x - v$. So $x = z + v$. But z and v are orthogonal.

$$\begin{aligned} \langle z, v \rangle &= \langle x - v, v \rangle = \langle x, v \rangle - \langle v, v \rangle \\ &= \langle x, u \rangle^2 - \langle x, u \rangle^2 \, \langle u, u \rangle = 0 \end{aligned}$$

Since

$$\| x + y \|^2 = (\| x \| + \| y \|)^2 = \| x \|^2 + 2 \, \| x \| \, \| y \| + \| y \|^2 ,$$

we have

$$\langle x, x \rangle + 2 \langle x, y \rangle + \langle y, y \rangle = \langle x, x \rangle + 2 \, \| x \| \, \| y \| + \langle y, y \rangle$$

and

$$\langle x, y \rangle = \| x \| \, \| y \| .$$

So $\langle x, u \rangle = \| x \|$.

$$\begin{aligned} \| x \|^2 &= \langle z + v, z + v \rangle = \langle z, z \rangle + 2 \langle z, v \rangle + \langle v, v \rangle = \| z \|^2 + \| v \|^2 \\ &= \| z \|^2 + \langle x, u \rangle^2 = \| z \|^2 + \| x \|^2 . \end{aligned}$$

We must have $\| z \| = 0$. So $z = 0$ and $x = v$ as claimed.

Notice that we have written the proof for a real inner product space. Exercise: what happens with a complex inner product ? ♦

◇ **1.6-5.** Find the equation of the line through $(1,1,1)$ and $(2,3,4)$. Is this line a linear subspace?

Answer. $x = (y+1)/2 = (z+2)/3$. Or $P(t) = (1+t, 1+2t, 1+3t)$. This line is not a linear subspace since $(0,0,0)$ is not on it. ◊

Solution. Let $A = (1,1,1)$ and $B = (2,3,4)$. The vector from A to B is $v = B - A = < 1, 2, 3 >$. For each real t, the point $P(t) = A + tv = (1+t, 1+2t, 1+3t)$ will lie on the line through A and B. This is a parameterization of that line. Note that $P(0) = A$ and $P(1) = B$. If t is between 0 and 1, then $P(t)$ is on the straight line segment between A and B.

Relations among x, y, and z along the line can be derived from the parameterization.

$$y = 1 + 2t = 1 + t + t = x + (x-1) = 2x - 1.$$

$$z = 1 + 3t = 1 + t + 2t = x + 2(x-1) = 3x - 2.$$

or

$$(y+1)/2 = x = (z+2)/3.$$

♦

◇ **1.7-4.** Using the inner product in Exercise 1.7-3, verify the triangle inequality for $f(x) = x$ and $g(x) = x^2$.

Solution. Direct computation gives

$$\begin{aligned}\| f+g \|^2 &= \int_0^1 (x+x^2)^2\,dx = \int_0^1 (x^2 + 2x^3 + x^4)\,dx \\ &= \frac{1}{3} + \frac{1}{2} + \frac{1}{5} = \frac{10+15+6}{30} = \frac{31}{30}.\end{aligned}$$

So $\| f+g \| = \sqrt{31/30}$. Also

$$\| f \|^2 = \int_0^1 x^2\,dx = \frac{1}{3} \qquad \text{and} \qquad \| g \|^2 = \int_0^1 x^4\,dx = \frac{1}{5}.$$

So $\| f \| + \| g \| = 1/\sqrt{3} + 1/\sqrt{5} = (\sqrt{5} + \sqrt{3})/\sqrt{15}$. The triangle inequality becomes

$$\sqrt{\frac{31}{30}} \le \frac{\sqrt{5}+\sqrt{3}}{\sqrt{15}} = \frac{\sqrt{10}+\sqrt{6}}{\sqrt{30}}$$

This is equivalent to $\sqrt{31} \le \sqrt{10} + \sqrt{6}$ or, squaring, to $31 \le 10 + 2\sqrt{60} + 6 = 16 + 4\sqrt{15}$. By subtracting 16, we see that this is equivalent to $15 \le 4\sqrt{15}$ which is equivalent to $\sqrt{15} \le 4$ or $15 \le 16$. This is certainly true, so the triangle inequality does indeed hold for these two functions. ♦

Exercises for Chapter 1

◇ **1E-1.** For each of the following sets S, find $\sup(S)$ and $\inf(S)$ if they exist:

(a) $\{x \in \mathbb{R} \mid x^2 < 5\}$
(b) $\{x \in \mathbb{R} \mid x^2 > 7\}$
(c) $\{1/n \mid n, \text{ an integer}, n > 0\}$
(d) $\{-1/n \mid n \text{ an integer}, n > 0\}$
(e) $\{.3, .33, .333, \dots\}$

Answer. (a) $\sup(S) = \sqrt{5}$; $\inf(S) = -\sqrt{5}$.
(b) Neither $\sup(S)$ nor $\inf(S)$ exist (except as $\pm\infty$).
(c) $\sup(S) = 1$; $\inf(S) = 0$.
(d) $\sup(S) = 0$; $\inf(S) = -1$.
(e) $\sup(S) = 1/3$; $\inf(S) = 0.3$.

◇

Solution. (a) $A = \{x \in \mathbb{R} \mid x^2 < 5\} =]-\sqrt{5}, \sqrt{5}[$, so $\inf A = -\sqrt{5}$ and $\sup A = \sqrt{5}$.
(b) $B = \{x \in \mathbb{R} \mid x^2 > 7\} =]-\infty, -\sqrt{7}[\cup]\sqrt{7}, \infty[$. The set is bounded neither above nor below, so $\sup B = \infty$, and $\inf B = -\infty$.

(c) $C = \{1/n \mid n \in \mathbb{Z}, n > 0\} = \{1, 1/2, 1/3, 1/4 \dots\}$. All elements are positive, so $\inf C \geq 0$, and they come arbitrarily close to 0, so $\inf C = 0$. The number 1 is in the set and nothing larger is in the set, so $\sup C = 1$.
(d) $D = \{-1/n \mid n \in \mathbb{Z}, n > 0\} = \{-1, -1/2, -1/3, -1/4 \dots\}$. All elements are negative, so $\sup D \leq 0$, and they come arbitrarily close to 0, so $\sup C = 0$. The number -1 is in the set and nothing smaller is in the set, so $\inf C = -1$.
(e) $E = \{0.3, 0.33, 0.333, \dots\}$. The number 0.3 is in the set and nothing smaller is in the set, so $\inf E = 0.3$. The elements of the set are all smaller than $1/3$, so $\sup E \leq 1/3$, and they come arbitrarily close to $1/3$, so $\sup E = 1/3$.

◆

◇ **1E-3.** (a) Let $x \geq 0$ be a real number such that for any $\varepsilon > 0$, $x \leq \varepsilon$. Show that $x = 0$.

(b) Let $S =]0, 1[$. Show that for each $\varepsilon > 0$ there exists an $x \in S$ such that $x < \varepsilon$.

Suggestion. (a) Suppose $x > 0$ and consider $\varepsilon = x/2$.

(b) Let $x = \min\{\varepsilon/2, 1/2\}$.

$\diamondsuit$

Solution. (a) Suppose $x \geq 0$ and that $x \leq \varepsilon$ for every $\varepsilon > 0$. If $x > 0$, then we would have $x/2 > 0$ so that $0 < x \leq x/2$ by hypothesis. Multiplication by the positive number 2 gives $0 \leq 2x \leq x$. Subtraction of x gives $-x \leq x \leq 0$. We are left with $x \geq 0$ and $x \leq 0$, so that $x = 0$ Thus the assumption that x is strictly greater than 0 fails, and we must have $x = 0$.

Notice that the argument given for part (a) shows that if $x \geq 0$, then $0 \leq x/2 \leq x$ and if $x > 0$, then $0 < x/2 < x$.

(b) Suppose $S =]0, 1[$, and $\varepsilon > 0$. If $\varepsilon \geq 1$, let $x = 1/2$. Then $x \in S$, and $x < \varepsilon$. If $\varepsilon < 1$, let $x = \varepsilon > 2$. Then $0 < x = \varepsilon/2 < \varepsilon < 1$. So $x \in S$, and $x < \varepsilon$. So in either case we have an x with $x \in S$, and $x < \varepsilon$.

$\blacklozenge$

$\diamond$ **1E-7.** For nonempty sets $A, B \subset \mathbb{R}$, let $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

Sketch. Since $\sup(A) + \sup(B)$ is an upper bound for $A + B$ (Why?), $\sup(A + B) \leq \sup(A) + \sup(B)$. Next show that elements of $A + B$ get arbitrarily close to $\sup(A) + \sup(B)$. $\diamondsuit$

Solution. Suppose $w \in A + B$. Then there are x in A and y in B with $w = x + y$. So

$$w = x + y \leq x + \sup B \leq \sup A + \sup B.$$

The number $\sup A + \sup B$ is an upper bound for the set $A + B$, so $\sup(A + B) \leq \sup A + \sup B$.

Now let $\varepsilon > 0$. Then there are x in A and y in B with

$$\sup A - \varepsilon/2 < x \quad \text{and} \quad \sup B - \varepsilon/2 < y.$$

Adding these inequalities gives

$$\sup A + \sup B - \varepsilon < x + y \in A + B.$$

So $\sup A + \sup B - \varepsilon$ is not an upper bound for the set $A + B$. Thus $\sup(A + B) > \sup A + \sup B - \varepsilon$. This is true for all $\varepsilon > 0$, so $\sup(A + B) \geq \sup A + \sup B$.

We have inequality in both directions, so $\sup(A + B) = \sup A + \sup B$ as claimed. $\blacklozenge$

◇ **1E-8.** For nonempty sets $A, B \subset \mathbb{R}$, determine which of the following statements are true. Prove the true statements and give a counterexample for those that are false:

(a) $\sup(A \cap B) \leq \inf\{\sup(A), \sup(B)\}$.
(b) $\sup(A \cap B) = \inf\{\sup(A), \sup(B)\}$.
(c) $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$.
(d) $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$.

Solution. (a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.

This is true if the intersection is not empty.
If $x \in A \cap B$, then $x \in A$, so $x \leq \sup A$. Also, $x \in B$, so $x \leq \sup B$. Thus x is no larger than the smaller of the two numbers $\sup A$, and $\sup B$. That is, $x \leq \inf\{\sup A, \sup B\}$ for every x in $A \cap B$. Thus $\inf\{\sup A, \sup B\}$ is an upper bound for $A \cap B$. So $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.
Here is another argument using Proposition 1.3.3. We notice that $\sup(A \cap B) \subseteq A$, so by Proposition 1.3.3 we should have $\sup(A \cap B) \leq \sup A$. Similarly, $\sup(A \cap B) \leq \sup B$. So $\sup(A \cap B)$ is no larger than the smaller of the two numbers $\sup A$ and $\sup B$. That is

$$\sup(A \cap B) \leq \inf\{\sup A, \sup B\}.$$

There is a problem if the intersection is empty. We have defined $\sup(\emptyset)$ to be $+\infty$, and this is likely to be larger than $\inf\{\sup A, \sup B\}$.
One can make a reasonable argument for defining $\sup(\emptyset) = -\infty$. Since any real number is an upper bound for the empty set, and the supremum is to be smaller than any other upper bound, we should take $\sup(\emptyset) = -\infty$. (Also, since any real number is a lower bound for $\emptyset$, and the infimum is to be larger than any other lower bound, we could set $\inf(\emptyset) = +\infty$). If we did this then the inequality would be true even if the intersection were empty.

(b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$
This one need not be true even if the intersection is not empty. Consider the two element sets $A = \{1, 2\}$ and $B = \{1, 3\}$, Then $A \cap B = \{1\}$. So $\sup(A \cap B) = 1$. But $\sup A = 2$ and $\sup B = 3$. So $\inf\{\sup A, \sup B\} = 2$ We do have $1 \leq 2$, but they are certainly not equal.
We know that $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$. Can we get the opposite inequality?

(c) $\sup(A \cup B) \geq \sup\{\sup A, \sup B\}$

(d) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$
Neither A nor B nor the union is empty, so we will not be troubled with problems in the definition of the supremum of the empty set.
If x is in $a \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \leq \sup A$. If $x \in B$, then $x \leq \sup B$. In either case it is no larger than the larger of the two numbers $\sup A$ and $\sup B$. So $x \leq \sup\{\sup A, \sup B\}$ for every x in the union. Thus $\sup\{\sup A, \sup B\}$ is an upper bound for $A \cup$. So

$$\sup(A \cup B) \leq \sup\{\sup A, \sup B\}$$

In the opposite direction, we note that $A \subseteq A \cup B$, so by Proposition 1.3.3, we have $\sup A \leq \sup(A \cup B)$. Similarly $\sup B \leq \sup(A \cup B)$. So $\sup(A \cup B)$ is at least as large as the larger of the two numbers $\sup A$ and $\sup B$. That is

$$\sup(A \cup B) \geq \sup\{\sup A, \sup B\}$$

We have inequality in both directions, so in fact equality must hold. ♦

◇ **1E-10.** Verify that the bounded metric in Example 1.7.2(d) is indeed a metric.

Sketch. Use the basic properties of a metric for $d(x, y)$ to establish those properties for $\rho(x, y)$. For the triangle inequality, work backwards from what you want to discover a proof. ◇

Solution. Suppose d is a metric on a set M, and ρ is defined by $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. We are to show that this is a metric on M and that it is bounded by 1.

(i) Positivity: Since $d(x, y) \geq 0$, we have $0 \leq d(x, y) < 1 + d(x, y)$. So

$$0 \leq \frac{d(x, y)}{1 + d(x, y)} < 1 \quad \text{for every } x \text{ and } y \text{ in } M.$$

Notice that this also shows that ρ is bounded by 1.

(ii) Nondegeneracy: If $x = y$, then $d(x, y) = 0$, so $\rho(x, y) = 0$ also. In the other direction, we know that $1 + d(x, y)$ is never 0, so

$$\begin{aligned} \rho(x, y) = 0 &\iff \frac{d(x, y)}{1 + d(x, y)} = 0 \\ &\iff d(x, y) = 0 \\ &\iff x = y \end{aligned}$$

(iii) Symmetry: Since $d(y, x) = d(x, y)$, we have

$$\rho(y, x) = \frac{d(y, x)}{1 + d(y, x)} = \frac{d(x, y)}{1 + d(x, y)} = \rho(x, y).$$

(iv) Triangle Inequality: Suppose x, y, and z are in M. Then

$$\begin{aligned} \rho(x, y) \leq \rho(x, z) + \rho(z, y) &\iff \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} \\ &\iff d(x, y)(1 + d(x, z))(1 + d(z, y)) \leq d(x, z)(1 + d(x, y))(1 + d(z, y)) \\ &\qquad + d(z, y)(1 + d(x, y))(1 + d(x, z)) \\ &\iff d(x, y) + d(x, y)d(x, z) + d(x, y)d(z, y) + d(x, y)d(x, z)d(z, y) \\ &\qquad \leq d(x, z) + d(x, z)d(x, y) + d(x, z)d(z, y) + d(x, y)d(x, z)d(z, y) \\ &\qquad + d(z, y) + d(z, y)d(x, y) + d(z, y)d(x, z) + d(x, y)d(x, z)d(z, y) \\ &\iff d(x, y) \leq d(x, z) + d(z, y) + 2(d(x, z)d(z, y) + d(x, y)d(x, z)d(z, y)) \end{aligned}$$

This last line is true since $d(x, y) \leq d(y, z) + d(z, y)$ by the triangle inequality for d and the last term is non-negative.

The function ρ satisfies all of the defining properties of a metric, so it is a metric on the space M.

◇ **1E-12.** In an inner product space show that

(a) $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ (***parallelogram law***).

(b) $\|x+y\| \; \|x-y\| \leq \|x\|^2 + \|y\|^2$.

(c) $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ (***polarization identity***).

Interpret these results geometrically in terms of the parallelogram formed by x and y.

Solution. In all of these, the key is to use the relationship between the inner product and the norm derived from it: $\|f\|^2 = \langle f, f\rangle$.

(a) We compute

$$\begin{aligned}
\| x+y \|^2 + \| x-y \|^2 &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\
&= \langle x+y, x \rangle + \langle x+y, y \rangle + \langle x-y, x \rangle + \langle x-y, -y \rangle \\
&= \langle x, x+y \rangle + \langle y, x+y \rangle + \langle x, x-y \rangle + \langle -y, x-y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad + \langle x, x \rangle + \langle x, -y \rangle + \langle -y, x \rangle + \langle -y, -y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad + \langle x, x \rangle + \langle x, -y \rangle - \langle y, x \rangle - \langle y, -y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad + \langle x, x \rangle + \langle -y, x \rangle - \langle y, x \rangle - \langle -y, y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad + \langle x, x \rangle - \langle y, x \rangle - \langle y, x \rangle + \langle y, y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\
&= \langle x, x \rangle + \langle y, y \rangle + \langle x, x \rangle + \langle y, y \rangle \\
&= 2 \| x \|^2 + 2 \| y \|^2
\end{aligned}$$

as desired.

(b) Using the result of part (a), we compute

$$\begin{aligned}
0 &\le (\| x+y \| - \| x-y \|)^2 \\
&\le \| x+y \|^2 - 2 \| x+y \| \, \| x-y \| + \| x-y \|^2 \\
&\le 2 \| x \|^2 - 2 \| x+y \| \, \| x-y \| + 2 \| y \|^2
\end{aligned}$$

So $\| x+y \| \, \| x-y \| \le \| x \|^2 + \| y \|^2$ as desired.

(c) The computation is like that of part (a), but with some sign changes.

$$\begin{aligned}
\| x+y \|^2 - \| x-y \|^2 &= \langle x+y, x+y \rangle - \langle x-y, x-y \rangle \\
&= \langle x+y, x \rangle + \langle x+y, y \rangle - \langle x-y, x \rangle - \langle x-y, -y \rangle \\
&= \langle x, x+y \rangle + \langle y, x+y \rangle - \langle x, x-y \rangle - \langle -y, x-y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad - \langle x, x \rangle - \langle x, -y \rangle - \langle -y, x \rangle - \langle -y, -y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad - \langle x, x \rangle - \langle x, -y \rangle + \langle y, x \rangle + \langle y, -y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad - \langle x, x \rangle - \langle -y, x \rangle + \langle y, x \rangle + \langle -y, y \rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&\qquad - \langle x, x \rangle + \langle y, x \rangle + \langle y, x \rangle - \langle y, y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle x, y \rangle + \langle y, y \rangle \\
&\qquad - \langle x, x \rangle + \langle x, y \rangle + \langle x, y \rangle - \langle y, y \rangle \\
&= 4 \langle x, y \rangle
\end{aligned}$$

as desired.

◇ **1E-15.** Let x_n be a sequence in $\mathbb{R}$ such that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)/2$. Show that x_n is a Cauchy sequence.

Sketch. First show that $d(x_n, x_{n+1}) \leq d(x_1, x_2)/2^{n-1}$, then that $d(x_n, x_{n+k})/2^{n-2}$. ◇

Solution. There is nothing particularly special about the number 1/2 in this problem. Any constant c with $0 \leq c < 1$ will do as well. The key is the convergence of the geometric series with ratio c. $\sum_{k=0}^{\infty} c^k = 1/(1-c)$.

Proposition. *If $\langle x_n \rangle_1^{\infty}$ is a sequence in $\mathbb{R}$ such that there is a constant c with $0 \leq c < 1$ and $d(x_n, x_{n+1}) \leq c d(x_{n-1}, x_n)$ for every index n, then $\langle x_n \rangle_1^{\infty}$ is a Cauchy sequence.*

Proof: First we show that $d(x_m, x_{m+1}) \leq c^{m-1} d(x_1, x_2)$ for each $m = 1, 2, 3, \ldots$. by an informal induction

$$\begin{aligned} d(x_m, x_{m+1}) &\leq c d(x_{m-1}, x_m) \\ &\leq c^2 d(x_{m-2}, x_{m-1}) \\ &\;\;\vdots \\ &\leq c^{m-1} d(x_1, x_2) \end{aligned}$$

Now we use this to establish that $\langle x_n \rangle_1^{\infty}$ is a Cauchy sequence in the form: For each $\varepsilon > 0$ there is an N such that $d(x_n, x_{n+p}) < \varepsilon$ whenever $n \geq N$ and $p > 0$.

The first step is a repeated use of the triangle inequality to get an expression in which we can use the inequality just established.

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots d(x_{n+p-1}, x_{n+p}) \\ &\leq \left(c^{n-1} + c^n + c^{n+1} + \cdots c^{n+p-2}\right) d(x_1, x_2) \\ &\leq c^{n-1}\left(1 + c + c^2 + \cdots + c^{p-1}\right) d(x_1, x_2) \\ &\leq \frac{c^{n-1}}{1-c} d(x_1, x_2). \end{aligned}$$

Since $0 \leq c < 1$, we know that $c^n \to 0$ as $n \to \infty$. If $\varepsilon > 0$, we can select N large enough so that $\frac{c^{N-1}}{1-c} d(x_1, x_2) < \varepsilon$. If $n \geq N$ and $p > 0$, we conclude that

$$d(x_n, x_{n+p}) \leq \frac{c^{n-1}}{1-c} d(x_1, x_2) \leq \frac{c^{N-1}}{1-c} d(x_1, x_2) < \varepsilon.$$

Thus $\langle x_n \rangle_1^{\infty}$ is a Cauchy sequence as claimed.

This solution has been written in terms of the metric $d(x, y)$ between x and y instead of $|x - y|$ to emphasize that it works perfectly well for a sequence $\langle x_n \rangle_1^{\infty}$ in any metric space provided there is a constant c satisfying the hypothesized inequality. In this form the exercise forms the key part of the proof of a very important theorem about complete metric spaces called the *Banach Fixed Point Theorem* or the *Contraction Mapping Principle*. We will study this theorem and some of its consequences in Chapter 5. ♦

◇ **1E-22.** (a) If x_n and y_n are bounded sequences in $\mathbb{R}$, prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

(b) Is the product rule true for lim sups?

Suggestion. Show that if $\varepsilon > 0$, then $x_n + y_n < \limsup x_n + \limsup y_n + \varepsilon$ for large enough n. ◇

Solution. (a) Let $A = \limsup x_n$ and $B = \limsup y_n$. Then A and B are both finite real numbers since $\langle x_n \rangle_1^\infty$ and $\langle y_n \rangle_1^\infty$ are bounded sequences. Suppose $\varepsilon > 0$. There are indices N_1 and N_2 such that $x_n < A + \varepsilon/2$ whenever $n \geq N_1$ and $y_n < B + \varepsilon/2$ whenever $n \geq N_2$. Let $N = \max(N_1, N_2)$. If $n \geq N$, we have

$$x_n + y_n < x_n + B + \varepsilon/2 < A + B + \varepsilon.$$

Since this can be done for any $\varepsilon > 0$, we conclude that $x_n + y_n$ can have no cluster points larger than A+B. Thus

$$\limsup(x_n + y_n) \leq A + B = \limsup x_n + \limsup y_n$$

as claimed.

We cannot guarantee equality. The inequality just established could be strict. For example, consider the sequence defined by $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then $\limsup x_n = \limsup y_n = 1$, but $x_n + y_n = 0$ for each n, so $\limsup(x_n + y_n) = 0$.

(b) There does not seem to be any reasonable relationship which can be guaranteed between the two quantities $\limsup(x_n y_n)$ and $(\limsup x_n) \cdot (\limsup y_n)$. Each of the possibilities "$<$", "$=$", and "$>$" can occur:
Example 1. Let $\langle x_n \rangle_1^\infty$ be the sequence $\langle 1, 0, 1, 0, 1, 0, \dots \rangle$ and $\langle y_n \rangle_1^\infty$ be the sequence $\langle 0, 1, 0, 1, 0, 1, \dots \rangle$. Then $\limsup x_n = \limsup y_n = 1$. But $x_n y_n = 0$ for all n. We have

$$\limsup(x_n y_n) = 0 < 1 = (\limsup x_n) \cdot (\limsup y_n).$$

Example 2. Let $x_n = y_n = 1$ for each n. Then $\limsup x_n = \limsup y_n = 1$. But $x_n y_n = 1$ for all n. We have

$$\limsup(x_n y_n) = 1 = (\limsup x_n) \cdot (\limsup y_n).$$

Example 3. Let $\langle x_n \rangle_1^\infty$ be the sequence $\langle 0, -1, 0, -1, 0, -1, 0, \dots \rangle$ and $\langle y_n \rangle_1^\infty$ be the sequence $\langle 0, -1, 0, -1, 0, -1, \dots \rangle$. Then $\limsup x_n = \limsup y_n = 0$. But $x_n y_n = 0$ is the sequence $\langle 0, 1, 0, 1, 0, 1, \dots \rangle$. We have

$$\limsup(x_n y_n) = 1 > 0 = (\limsup x_n) \cdot (\limsup y_n).$$

◆

◇ **1E-25.** We say that $P \le Q$ if for each $x \in P$, there is a $y \in Q$ with $x \le y$.

(a) If $P \le Q$, then show that $\sup(P) \le \sup(Q)$.
(b) Is it true that $\inf(P) \le \inf(Q)$?
(c) If $P \le Q$ and $Q \le P$, does $P = Q$?

Solution. (a) If $x \in P$, then there is a $y \in Q$ with $x \le y \le \sup Q$. So $\sup Q$ is an upper bound for P. Thus $\sup P \le \sup Q$ as claimed.

(b) Not necessarily. It is possible that $\inf P > \inf Q$. Consider $P = [0, 1]$ and $Q = [-1, 1]$. Then $P \le Q$ but $\inf P > \inf Q$.

(c) Again, not necessarily. It is possible that P and Q are different sets even if $P \le Q$ and $Q \le P$. Consider, for example, $P = \mathbb{R} \setminus \mathbb{Q}$, and $Q = \mathbb{Q}$. Here is another: $P = \{1\}$ and $Q = \{0, 1\}$. Then $P \le Q$ and $Q \le P$, but $P \ne Q$.

♦

◇ **1E-26.** Assume that $A = \{a_{m,n} \mid m = 1, 2, 3, \dots \text{ and } n = 1, 2, 3, \dots\}$ is a bounded set and that $a_{m,n} \ge a_{p,q}$ whenever $m \ge p$ and $n \ge q$. Show that

$$\lim_{n\to\infty} a_{n,n} = \sup A.$$

Solution. Since A is a bounded, nonempty subset of $\mathbb{R}$, we know that $c = \sup A$ exists as a finite real number and that $a_{j,k} \le c$ for all j and k. For convenience, let $b_n = a_{n,n}$. If $n \le k$, we have $b_n = a_{n,n} \le a_{k,k} = b_k$. So the sequence $\langle b_n \rangle_1^\infty$ is increasing and bounded above by c. So $b = \lim_{n\to\infty} b_n$ exists and $b \le c$. Let $d < c$, then there is a $a_{k,j}$ in A with $d < a_{k,j} \le c$. If $n \ge \max(k, j)$, then

$$d < a_{k,j} \le a_{n,n} = b_n \le b_{n+1} \le \dots .$$

So $b = \lim b_n > d$. This is true for every $d < c$, so $b \ge c$. We have inequality in both directions, so

$$\lim_{n\to\infty} a_{n,n} = \lim_{n\to\infty} b_n = b = c = \sup A$$

as claimed. ♦

◇ **1E-28.** Let x_n be a convergent sequence in $\mathbb{R}$ and define $A_n = \sup\{x_n, x_{n+1}, \dots\}$ and $B_n = \inf\{x_n, x_{n+1}, \dots\}$. Prove that A_n converges to the same limit as B_n, which in turn is the same as the limit of x_n.

Solution. Suppose $x_n \to a \in \mathbb{R}$. Let $\varepsilon > 0$. There is an index N such that $a-\varepsilon < x_k < a+\varepsilon$ whenever $k \geq N$. Thus if $n \geq N$ we have $a-\varepsilon < x < a+\varepsilon$ for all x in the set $S_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$. That is, $a-\varepsilon$ is a lower bound for S_n, and $a+\varepsilon$ is an upper bound. Hence

$$a - \varepsilon \leq B_n = \inf S_n \leq \sup S_n = A_n \leq a + \varepsilon.$$

So

$$|A_n - a| \leq \varepsilon \qquad \text{and} \qquad |B_n - a| \leq \varepsilon$$

whenever $n \geq N$. We conclude that

$$\lim_{n\to\infty} A_n = a = \lim_{n\to\infty} B_n$$

as claimed. ♦

◇ **1E-30.** Let $\mathcal{V}$ be the vector space $\mathcal{C}([0,1])$ with the norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0,1]\}$. Show that the parallelogram law fails and conclude that this norm does not come from any inner product. (Refer to Exercise 1E-12.)

Solution. Let $f(x) = x$ and $g(x) = 1 - x$. Both of these functions are in $\mathcal{C}([0,1])$, and $\|f\|_\infty = \sup\{x \mid x \in [0,1]\} = 1$ and $\|g\|_\infty = \sup\{1 - x \mid x \in [0,1]\} = 1$. For the sum and difference we have, $(f+g)(x) = 1$ and $(f-g)(x) = 2x - 1$. So

$$\|f+g\|_\infty + \|f-g\|_\infty = 1 + 1 = 2$$

while

$$2\|f\|_\infty^2 + 2\|g\|_\infty^2 = 2 \cdot 1^2 + 2 \cdot 1^2 = 4.$$

Since 2 and 4 are not the same, we see that this norm does not satisfy the parallelogram law. If there were any way to define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{C}([0,1])$ in such a way that $\|h\|_\infty^2 = \langle h, h \rangle$ for each h in the space, then the parallelogram law would have to hold by the work of Exercise 1E-12(a). Since it does not, there can be no such inner product.

Another example which is useful for other problems also is shown in the figure. We let f and g be "tent functions" based on intervals which do not overlap. For example we can use the subintervals $[0,1/2]$ and $[1/2,1]$ of the unit interval as illustrated in the figure.

Ignore the figure!

For these choices of f and g, we have

$$\|f\|_\infty = \|g\|_\infty = \|f+g\|_\infty = \|f-g\|_\infty = 1.$$

So

$$\|f+g\|_\infty{}^2 + \|f-g\|_\infty{}^2 = 2,$$

while

$$2\|f\|_\infty{}^2 + 2\|g\|_\infty{}^2 = 4.$$

Since 2 is not equal to 4, the parallelogram law fails for these functions also. ♦